2019 GRWC PROBLEM PROPOSAL

Guangyi Yue

The main problem is the following conjecture of Bezrukavnikov, which relates two series of operations B^1_I, B^2_I, and their indicator functions D^1_I, D^2_I, on partitions.

This problem comes from an ongoing project of Bezrukavnikov and Okounkov, where they extend the method of relating the representation of the Lie algebra of a semi-simple algebraic group and the generalized Springer fiber in [BMR08] to representation of symplectic reflection algebras in type A, which is closely related to the quantum cohomology of the Hilbert scheme in [OP10]. Here we only consider the relevant combinatorics.

Let $\mathcal{P}_n$ denote the collection of all integer partitions of n, and let $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$. For any partition $\lambda \vdash n$, $|\lambda| = n$ denotes its size, and λ^{T} denotes its transpose (or conjugate).

For a fixed positive integer n, the *Farey sequence* F_n is the set of reduced fractions $0 \leq \frac{a}{b} \leq 1$ with denominator at most n. Each $\frac{a}{b} \in F_n$ is called a *wall*. Let $\mathcal{I}$ be the set of sub-intervals of $[0, 1]$ whose endpoints are consecutive walls.

Definition 0.1. We define a collection of maps $B^1_I : \mathcal{P}_n \to \mathcal{P}_n$ inductively as follows. First define $B^1_{[0,\frac{1}{n}]}$ to be the identity map. Suppose we have defined B^1_I for $I = \left[\frac{a_{i-1}}{b_{i-1}}, \frac{a_i}{b_i}\right] \in \mathcal{I}$, and suppose we have the unique decomposition $B^1_I(\lambda) = \mu \cup b_i\nu$, where μ consists of the parts of λ not divisible by b_i, $b_i\nu = (b_i\nu_1, \dots)$ and $\cup$ denotes the union of partitions as multi-sets. Then for the adjacent interval $I' = \left[\frac{a_i}{b_i}, \frac{a_{i+1}}{b_{i+1}}\right] \in \mathcal{I}$, we define

$$B^1_{I'}(\lambda) = \mu \cup b_i(\nu^{\mathrm{T}}) \qquad \text{and} \qquad D^1_I(\lambda) = b_i \cdot |\nu|.$$

The second series of operations, which are more convoluted, are compositions of the extended Mullineux transpose $\mathcal{W}_b : \mathcal{P} \to \mathcal{P}$.

Let $m_i = m_i(\lambda)$ denote the multiplicity of the part i in the partition λ, so $\lambda = 1^{m_1}2^{m_2}\cdots$. For a fixed positive integer b, we can uniquely decompose each $m_i = bq_i + r_i$ for some positive integers q_i, r_i such that $0 \leq r_i < b$. The *regular part of* λ is $\mathrm{Reg}_b(\lambda) = 1^{r_1}2^{r_2}\dots$ and the *irregular part of* λ is $\mathrm{Irr}_b(\lambda) = (1^{q_1}2^{q_2}\dots)$. Hence $\lambda = \mathrm{Reg}_b(\lambda) \cup b\star\mathrm{Irr}_b(\lambda)$ where the operator $b\star$ indicates repeating each part of the partition b times. λ is called *b-regular* if $\mathrm{Irr}_b(\lambda) = \emptyset$.

It is well-known that any irreducible p-modular representation $\rho = \rho_\lambda$ of the symmetric group $\mathfrak{S}_n$ are labeled uniquely by some p-regular partition $\lambda \vdash n$. The *Mullineux Involution* M_p is defined by $\rho_{\lambda^{\mathrm{M}_p}} = \rho_\lambda \otimes \mathrm{sgn}$ where sgn is the sign representation. There are a few combinatorial ways to extend the definition to p being not necessarily prime, see [Kle96, FK97] for details.

The *extended Mullineux transpose* transformation $\mathcal{W}_b : \mathcal{P} \to \mathcal{P}$ is defined to be

$$\lambda^{\mathcal{W}_b} := \left(\mathrm{Reg}_b(\lambda)^{\mathrm{M}_b} \cup b \star (\mathrm{Irr}_b(\lambda)^{\mathrm{T}})\right)^{\mathrm{T}}.$$

In particular, if λ is b-regular, then $\lambda^{\mathcal{W}_b} = (\lambda^{\mathrm{M}_b})^{\mathrm{T}}$.

Definition 0.2. We define a collection of maps $B^2_I : \mathcal{P}_n \to \mathcal{P}_n$ inductively as follows. First define $B^2_{[0,\frac{1}{n}]}$ to be the identity map. Suppose we have defined B^2_I for $I = \left[\frac{a_{i-1}}{b_{i-1}}, \frac{a_i}{b_i}\right] \in \mathcal{I}$. Then for the adjacent interval $I' = \left[\frac{a_i}{b_i}, \frac{a_{i+1}}{b_{i+1}}\right] \in \mathcal{I}$, we define

$$B^2_{I'}(\lambda) = B^2_I(\lambda)^{\mathcal{W}_{b_i}} \qquad \text{and} \qquad D^2_I(\lambda) = b_i \cdot |\,\mathrm{Irr}_{b_i}(B^2_I(\lambda))|.$$

Conjecture 0.3 (Bezrukavnikov)**.** *For every partition $\lambda \vdash n$ and $I \in \mathcal{I}$, there is $D^1_I(\lambda) = D^2_I(\lambda^{\mathrm{T}})$.*

Example 0.4. *We give the example of $\lambda = (3, 2)$ in the following table.*

Interval	$[0, \frac{1}{5}]$	$[\frac{1}{5}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{3}]$	$[\frac{1}{3}, \frac{2}{5}]$	$[\frac{2}{5}, \frac{1}{2}]$	$[\frac{1}{2}, \frac{3}{5}]$	$[\frac{3}{5}, \frac{2}{3}]$	$[\frac{2}{3}, \frac{3}{4}]$	$[\frac{3}{4}, \frac{4}{5}]$	$[\frac{4}{5}, 1]$
B_I^1	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$	$(3,2)$
D_I^1	*0*	*0*	*3*	*0*	*2*	*0*	*3*	*0*	*0*	-
B_I^2	$(2^2,1)$	$(2^2,1)$	(1^5)	$(4,1)$	$(3,1^2)$	$(3,1^2)$	$(2,1^3)$	(5)	$(3,2)$	$(3,2)$
D_I^2	*0*	*0*	*3*	*0*	*2*	*0*	*3*	*0*	*0*	-

This big conjecture, still unsolved, has been studied in the following aspects:

(1) In [DY18], the second algorithm is shown to be identical to a series of generalized column regularizations when starting at the one-row partition (n). There are other nice properties when starting at (n): they always stay regular at each step, the quotients are always rectangles and the sequence of partitions is monotone. This result doesn't give much information for the numerical function $D_I^2((n)) = 0$ since $D_I^1((1^n)) = 0$ for all I as well. However, one observation is that when starting at other partitions, the resulting series is piecewise monotone, and the breaks are exactly those endpoints of I when $D_I^2 \neq 0$. So we need to study where those breaks are and between the breaks, whether we could still simplify the complicated wall-crossing (i.e. extended Mullineux transpose) as column regularization.

(2) Since the Mullineux map is very complicated in terms of combinatorics, the preprint [WY18] studies the condition under which one can use the two parameter column regularization $\mathrm{Colreg}_{a,b}$ to replace it. The conditions for $\lambda^{\mathrm{M}_b\,\mathrm{T}} = \lambda^{\mathrm{Colreg}_{a,b}}$ are precisely the following:
   - $\lambda^{\mathrm{Colreg}_{a,b}} \in \mathcal{P}$;
   - Let $l_{i,j}, a_{i,j}, H_{i,j}$ be the leg length, arm length, hook length for the box (i,j). For every $(i,j) \in \lambda$ satisfying $b \mid H_{ij}$, we have the following inequality (we call such a partition (a,b)-shallow):

$$\left(\frac{b}{a} - 1\right) l_{ij} < a_{ij} + 1$$

There are a few related questions:

(1) The precise definition of the two-parameter column regularization $\mathrm{Colreg}_{a,b}$ is given in [WY18]. Since λ is b-regular iff $\lambda^{\mathrm{Colreg}_{1,b}} = \lambda$, we could try to enumerate the (a,b)-regular partitions, i.e. λ satisfying $\lambda^{\mathrm{Colreg}_{a,b}} = \lambda$. For example, is there a nice generating function for (a,b)-regular partitions?

(2) If $\lambda^{\mathrm{M}_b\,\mathrm{T}} = \lambda^{\mathrm{Colreg}_{a,b}}$, is λ always (a,b)-shallow? This is the reverse direction of the result in [WY18]. [BOX99] proved this in case of $a = 1$.

References

[BMR08] Roman Bezrukavnikov, Ivan Mirković, and Dmitriy Rumynin. Localization of modules for a semisimple Lie algebra in prime characteristic. *Ann. of Math. (2)*, 167(3):945–991, 2008. With an appendix by Bezrukavnikov and Simon Riche.

[BOX99] Christine Bessenrodt, Jørn B Olsson, and Maozhi Xu. On properties of the Mullineux map with an application to Schur modules. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 126, pages 443–459. Cambridge University Press, 1999.

[DY18] Panagiotis Dimakis and Guangyi Yue. Combinatorial wall-crossing and the Mullineux involution. *Journal of Algebraic Combinatorics*, Sep 2018.

[FK97] Ben Ford and Alexander S Kleshchev. A proof of the Mullineux conjecture. *Mathematische Zeitschrift*, 226(2):267–308, 1997.

[Kle96] Alexander S Kleshchev. Branching rules for modular representations of symmetric groups III: some corollaries and a problem of Mullineux. *Journal of the London Mathematical Society*, 54(1):25–38, 1996.

[OP10] Andrei Okounkov and Rahul Pandharipande. Quantum cohomology of the hilbert scheme of points in the plane. *Inventiones mathematicae*, 179(3):523–557, 2010.

[WY18] Allen Wang and Guangyi Yue. Mullineux involution and the generalized regularization, i. *arXiv preprint arXiv:1812.07732*, 2018.