Proposed Research Problem

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Schur functions are important as they arise naturally in representation theory and geometry. For each partition λ, the Schur function s_λ can be defined combinatorially as generating functions for objects called *semistandard Young tableaux*.

In the study of symmetric functions and combinatorial representation theory, there are two important families of coefficients, Littlewood–Richardson (LR) coefficients $c^{\lambda}_{\mu,\nu}$ and Kronecker coefficients $g^{\lambda}_{\mu,\nu}$, which are the Schur structure constants of the usual product (denoted by $\cdot$) and Kronecker product (denoted by $*$) on symmetric functions respectively. That is,

$$s_\mu \cdot s_\nu = \sum_{\lambda} c^{\lambda}_{\mu,\nu} s_\lambda \quad \text{and} \quad s_\mu * s_\nu = \sum_{\alpha} g^{\lambda}_{\mu,\nu} s_\lambda.$$

There are several beautiful combinatorial descriptions (LR tableaux, Gelfand–Tsetlin patterns, hives, etc) for the LR coefficients (see [5, 6, 8]). We introduce one of them below.

A *Young tableau* is a filling of a Young diagram which assigns a positive integer to each box of the diagram. A Young tableau is called *semistandard* if the fillings is weakly increasing along the rows and strictly increasing down the columns, and such tableaux are called semistandard Young tableaux (SSYT). For a (skew) Young tableau T, we denote by $r(T)$ the *reading word* of T, which is obtained by reading the numbers in T from right to left in successive rows, starting with the top row. A word $a = a_1a_2\dots a_N$ ($a_i \in [n]$) is said to be a *lattice permutation* if for all $1 \le i \le n-1$ and $1 \le r \le N$, the number of occurrences of i in $a_1a_2\dots a_r$ is not less than the number of occurrences of $i+1$. For example, the following tableau T is semistandard and the reading word of the T is $r(T) = 11213224$, which is a lattice permutation.

$$\mathrm{T} = \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ \boxed{2} & \boxed{2} & \boxed{3} & \\ \boxed{4} & & & \end{array}$$

Proposition 1 (Littlewood–Richardson Rule)**.** *Let λ, μ, and ν be partitions. The Littlewood–Richardson coefficient $c^{\lambda}_{\mu,\nu}$ counts the number of semistandard skew Young tableaux T with shape λ/μ and weight ν such that the reading word $r(T)$ is a lattice permutation.*

While the Littlewood–Richardson coefficients are well-studied and have several beautiful combinatorial interpretations, an explicit combinatorial or geometric description for the Kronecker coefficients is still unknown except for some special cases.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ be a partition. A sequence $a = (a_1, a_2, \dots, a_n)$ is called an *α-lattice permutation* if the concatenation $(1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}a)$ is a lattice permutation. Let λ, α, and ν be partitions such that $\alpha \subset \lambda \cap \nu$ and let T be a semistandard skew tableau of shape λ/α and content $(\nu - \alpha)$. We call T a *Kronecker tableau* iff its reading word is an α-lattice permutation and one of the following three conditions is satisfied:

(1) $\alpha_1 = \alpha_2$.
(2) $\alpha_1 > \alpha_2$ and the number of 1's in the second row of λ/α is $\alpha_1 - \alpha_2$.
(3) $\alpha_1 > \alpha_2$ and the number of 2's in the first row of λ/α is $\alpha_1 - \alpha_2$.

We denote the number of Kronecker tableaux satisfying the above conditions by $K^{\lambda}_{\alpha,\mu}$. Ballantine and Orellana [2] showed

Proposition 2. *Let n and p be positive integers such that $n \geq 2p$ and λ be a partition of n. If $\lambda_1 \geq 2p-1$, then*

$$s_{(n-p,p)} * s_{\lambda} = \sum_{\nu} \sum_{\alpha \vdash p} K^{\lambda}_{\alpha,\nu}\; s_{\nu}.$$

There are also combinatorial formulas for the Kronecker product of two Schur functions when one of the index partitions has hook shape.

Aguiar, Ferrer Santos, and Moreira [1] introduced a new (commutative and associative) product, the Heisenberg product (denoted by #), on representation of symmetric groups, hence also on symmetric functions. This product interpolates between the usual product and the Kronecker product, so its Schur structure constants, called Heisenberg coefficients and denoted by $h^{\lambda}_{\mu,\nu}$, generalize both Littlewood–Richardson coefficients and Kronecker coefficients. Let μ and ν be partitions of n and m respectively, then

$$s_{\mu} \# s_{\nu} = \sum_{\ell=\max(n,m)}^{n+m} \sum_{\lambda \vdash \ell} h^{\lambda}_{\mu,\nu}\; s_{\lambda}.$$

When $\ell = m+n$, $h^{\lambda}_{\mu,\nu} = c^{\lambda}_{\mu,\nu}$; when $\ell = n = m$, $h^{\lambda}_{\mu,\nu} = g^{\lambda}_{\mu,\nu}$.

Question 1: Can we find combinatorial formulas for the Heisenberg product $s_{\mu} \# s_{\nu}$ when μ is a two-rows partition or a hook shape partition.

For a partition α, let $\square_{k,n}(\alpha)$ be the complement of α with respect to a $n \times k$ rectangle. Briand et al. [3] showed that the Kronecker coefficients and the LR coefficients share symmetries related to this complementation operation:

$$c^{\lambda}_{\mu,\nu} = c^{\square_{\ell+m,\,n}(\lambda)}_{\square_{\ell,\,n}(\mu),\,\square_{m,\,n}(\nu)} \quad \text{and} \quad g^{\lambda}_{\mu,\nu} = g^{\square_{n,\,\ell m}(\lambda)}_{\square_{mn,\,\ell}(\mu),\,\square_{\ell n,\,m}(\nu)}.$$

Heisenberg coefficients also have this symmetry

$$h^{\lambda}_{\mu,\nu} = h^{\square_{\ell,\,mn+m+n}(\lambda)}_{\square_{\ell n+\ell,\,m}(\mu),\,\square_{\ell m+\ell,\,n}(\nu)}.$$

A function $f : \mathbb{N} \to \mathbb{N}$ is a quasi-polynomial if there is a positive integer p and polynomials $q_0, \ldots, q_{p-1}$ such that $f(n) = q_i(n)$ when $n \equiv i \bmod p$. We know that the Littlewood–Richardson coefficient and the Kronecker coefficient have polynomiality [4, 9] and quasi-polynomiality [7] respectively. We treat partitions as vectors, then we can define addition and scalar multiplication on partitions. Given partitions λ, μ, and ν with appropriate sizes, the coefficients $c^{n\lambda}_{n\mu,n\nu}$ and $g^{n\lambda}_{n\mu,n\nu}$ are polynomial and quasi-polynomial in n respectively.

Question 2: Does the Heisenberg coefficient have quasi-polynomiality?

The triples of partitions of nonvanishing Heisenberg coefficients form a semigroup (i.e. if $h^{\lambda}_{\mu,\nu}$ and $h^{\alpha}_{\beta,\gamma}$ are nonzero, then $h^{\lambda+\alpha}_{\mu+\beta,\nu+\gamma}$ is also nonzero). This suggests us to consider the cone, called the Heisenberg cone, generated by this semigroup.

Question 3: What is the structure of the Heisenberg cone?

The Littlewood–Richardson cone and the Kronecker cone sit naturally inside the Heisenberg cone. It would be interesting to explore more relations among these three cones.

References

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