2019 GRWC Problem Statement

Antimagic orientations of graphs—Donglei Yang

1 Background

An *antimagic labeling* of a graph G is a bijection from $E(G)$ to $\{1, 2, \ldots, e(G)\}$ such that for any distinct vertices u and v, the sum of labels on edges incident with u differs from that for edges incident with v. A graph G is *antimagic* if it has an antimagic labeling. Hartsfield and Ringel [6] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic. The most significant progress on this problem is a result of Alon, Kaplan, Lev, Roditty, and Yuster [1], which states that there exists an absolute constant c such that every graph on n vertices with minimum degree at least $c \log n$ is antimagic. Eccles [4] improved this by showing that there exists an absolute constant c_0 such that if G is a graph with average degree at least c_0, and G contains no isolated edge and at most one isolated vertex, then G is antimagic. For regular graphs, Cranston, Liang, and Zhu [3] proved that every odd regular graph is antimagic, and later Chang, Liang, Pan, and Zhu [2] affirmed the even case.

Motivated by antimagic labelings of graphs, Hefetz, Mütze, and Schwartz [7] initiated the study of antimagic labelings of digraphs. Let D be a digraph. We use $A(D)$ and $V(D)$ to denote the set of arcs and vertices of D, respectively. A labeling of D from $A(D)$ to $\{1, 2, \ldots, |A(D)|\}$ is *antimagic* if no two vertices in D have the same vertex-sum, where the vertex-sum of a vertex $u \in V(D)$ is the sum of labels of all arcs entering u minus the sum of labels of all arcs leaving u. A digraph D is *antimagic* if it has an antimagic labeling. A graph G has an *antimagic orientation* if an orientation of G is antimagic.

Hefetz, Mütze, and Schwartz [7] raised the questions " Is every orientation of any connected graph antimagic?" and "Does every graph admit an antimagic orientation?". They proved an analogous result of Alon, Kaplan, Lev, Roditty, and Yuster [1] that there exists an absolute constant c such that every orientation of any graph on n vertices with minimum degree at least $c \log n$ is antimagic. Except for $K_{1,2}$ and K_3, no other counterexample to the first question is known. For the second question, they proposed the following conjecture.

Conjecture 1.1 ([7]). *Every connected graph admits an antimagic orientation.*

They gave the following results on certain classes of regular graphs.

Theorem 1.1. [7] *For any integer $d \geq 1$,*

(a) *every $(2d-1)$-regular graph admits an antimagic orientation;*

(b) *every connected, $2d$-regular graph G admits an antimagic orientation if G has a matching that covers all but at most one vertex of G.*

They also pointed out that "It seems hard to discard any of the two conditions in Theorem 1.1(b), that is connectedness and having a matching that covers all vertices but at most one. In fact, we do not even know if every disjoint union of cycles admits an antimagic orientation."

Recently, Li et al. [8] confirmed the case when G is a disjoint union of cycles and gave a stronger result than Theorem 1.1(b).

Theorem 1.2. [8] *For any integer $d \geq 2$,*

(a) *every 2-regular graph admits an antimagic orientation;*

(b) *every $2d$-regular graph with at most two odd components admits an antimagic orientation.*

As in [8], they put forward the following stronger conjecture.

Conjecture 1.2 ([8])**.** *Every graph admits an antimagic orientation.*

To add the evidence for the conjecture, we proved the following result.

Theorem 1.3 ([12])**.** *Every $2d$-regular graph G admits an antimagic orientation, where $d \geq 2$.*

Together with Theorem 1.1 (a) and Theorem 1.2, Theorem 1.3 implies that Conjecture 1.3 holds for all regular graphs.

Corollary 1.1. *Every regular graph admits an antimagic orientation.*

It is worth noting that the proof of Thereom 1.3 is similar to the proof of Theorem 1.2 given in [8]. We label the edges of odd components judiciously so that we obtain a desired antimagic orientation.

Applying the same arguments in the proof of Theorem 1.3, we can prove a more general result.

Corollary 1.2. *Let $d \geq 2$ be an integer. If every vertex of a graph G has degree $2d$ or $2d+2$, then G admits an antimagic orientation.*

Also, Shan and Yu [11] affirmed Conjecture 1.3 for biregular bipartite graphs. Except for the aforementioned classes of graphs, Conjecture 1.1 is widely open, even for trees. Moreover, it seems not easy to prove that G admits an antimagic orientation on condition that each component has an antimagic orientation.

In the following subsections, we think about some specific problems on this topic.

1.1 Eulerian graphs

A connected graph admits an Euler tour if and only if every vertex has even degree. An Euler tour plays a crucial role in the proof of Theorem 1.1. As a generalization of regular graphs, we consider the following problem.

Question 1. *Does every Eulerian graph admit an antimagic orientation?*

Given a Eulerian graph G and an Euler tour T, we observe that each vertex $v_i \in V(G)$ has d_i copies in T with $2d_i = d_G(v_i)$. When all d_i are almost equal, the method in the proof of Theorem 1.1 still works (see Corollary 1.2). When all the d_i's vary in a large scale, we need some new strategies.

1.2 Trees

Ringel and Hartsfield [6] made the following conjecture.

Conjecture 1.3. *Every tree on at least* 3 *vertices is antimagic.*

While some special classes of trees are confirmed antimagic [9, 10], this conjecture is still open. An easy observation is that, every 2-chromatic graph which is antimagic, admits an antimagic orientation. So any tree that is proved to be antimagic also admits an antimagic orientation. Therefore we consider the following question, which seems more flexible to deal with than Conjecture 1.3.

Question 2. *Does any tree admit an antimagic orientation?*

References

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