# $P$-matchings in graphs

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## 1 Background

Let $G$ be a graph. Then a matching $M$ of $G$ is a collection of vertex-disjoint edges. The matching number of $G$ is $\beta_1(G) = \max\{|M| : M \text{ is a matching of } G\}$. Defined similarly, the lower matching number of $G$ is $\beta_1^-(G) = \min\{|M| : M \text{ is a maximal matching}\}$. A matching $M$ is said to be perfect if it saturates all of the vertices of $G$. That is, every vertex is incident to an edge in $M$. There has already been a significant amount of work done in studying matchings, and in particular, perfect matchings [4]. One famous result dealing with the matching number of a graph $G$ is one of the Gallai identities.

**Theorem 1.** *(Gallai, 1959). If G has no isolated vertices, then*

$$\alpha_1(G) + \beta_1(G) = |V(G)|$$

*where $\alpha_1(G)$ is the smallest collection $S$ of edges such that every vertex in G is incident to an edge in $S$.*

Given a property $P$, we say that a matching $M$ is a $P$-matching if the subgraph of $G$ induced by the vertices included in $M$, $\langle M \rangle$, has property $P$ [2]. One can then consider $\beta_P(G) = \max\{|M| : \langle M \rangle \text{ has property } P\}$ and $\beta_P^- = \min\{|M| : \langle M \rangle \text{ is maximal with respect to } P\}$. For example, we would say that a matching $M$ is connected if the induced subgraph of $M$ is connected. We use $\beta_c$ and $\beta_c^-$ for the matching number and lower matching number respectively.

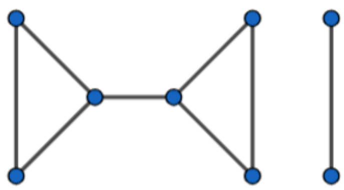

Figure 1: $\beta_1(G) = 4,\ \beta_1^-(G) = 3,\ \beta_c(G) = 3,\ \beta_c^-(G) = 1$

In the above example we have that $\beta_1(G) > \beta_c(G)$ and $\beta_1(G) > \beta_1^-(G)$, but these strict inequalities aren't always the case. For example, $\beta_1(K_n) = \beta_1^-(K_n)$ for all $n$, and it is shown in [3] that $\beta_1(G) = \beta_c(G)$ whenever $G$ is connected.

In [3], they consider several types of matchings including connected matchings, acyclic matchings, and disconnected matchings where the graph induced by the matching has the respective property. Their results include inequalities relating various types of matchings as well as results on complexity. One of their results is the following

**Theorem 2.** *For any tree $T$,*

$$\beta_1(T) - 1 \leq \beta_{dc}(T) \leq \beta_1(T)$$

*where $\beta_{dc}$ is the disconnected matching number.*

In [2], the authors introduce several new classes of matchings such as vertex-irredundant, independent, and bipartite matchings. A matching $M$ is said to be vertex-irredundant if for every edge $uv \in M$, either $u$ or $v$ has a neighbor $w$ such that $w$ is not adjacent to any other vertex in the subgraph induced by $M$. A matching is independent if it has an orientation $(X, Y)$ such that $X$ is independent. That is, a partition of the vertices used in the matching in which one side is independent in the induced subgraph. A matching is bipartite if it has an orientation $(X, Y)$ such that both $X$ and $Y$ are independent.

## 2 Proposed problems

Fenstermacher, Ganguly, Hedetniemi, and Laskar ended their paper [2] by proposing several problems related to these classes of matchings.

**Problem 1.** *For any of the types of $P$-matchings mentioned earlier, find a nontrivial upper bound for $\beta_P$ and lower bound for $\beta_P^-$.*

Other types of $P$-matchings that may be worth considering are planar matchings, matchings in which the induced subgraph is planar, or $k$-chromatic matchings. That is, we could consider matchings where the induced subgraph is $k$-colorable. As far as planar matchings, go $\beta_{pl}(G) = \beta_1(G)$ for any planar graph, but we could work our way up by looking at graphs with crossing number of 1 and to try to extend this.

**Problem 2.** *For a given $P$-matching, find a way to describe when a given matching $M$ is a maximum $P$-matching.*

Results about complexity are of interest as well. [3] lists several known results related matching sizes and complexity. The following problem looks to find an analogous result for the previously stated theorem by Gallai.

**Problem 3.** *Given a property $P$, find a parameter $\alpha_P$ such that $\beta_P(G) + \alpha_P(G) = |V(G)|$.*

It's unclear whether such a parameter naturally arises, but looking at small examples may provide insight into the structure of matchings with certain properties that could be useful. Finally, we end with a problem of looking for a Nordhaus-Gaddum type result. For results of a similar style see [1].

**Problem 4.** *For a given $P$-matching, find bounds $\gamma(n)$ and $\eta(n)$ such that*

$$\gamma \leq \beta_P(G) + \beta_P(\overline{G}) \leq \eta$$

*for any $n$ vertex graph $G$. Here $\overline{G}$ denotes the complement of $G$.*

## 3 References


1. M. Aouchiche, P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Applied Mathematics.* **161** (2013), 466-546.

2. T. Fenstermacher, S. Ganguly, S. T. Hedetniemi, and R. Laskar, $P$-matchings in graphs: A brief survey with some open problems, arXiv:1805.03334.

3. W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, and R. Laskar, Generalized subgraph-restricted matchings in graphs, *Discrete Mathematics.* **293(1-3)** (2005), 129-138.

4. L. Lovász and M. D. Plummer, *Matching Theory*, volume 29 of *Ann. Discrete Math.* North-Holland Publishing Co., Amsterdam (1986).