Graded Posets and Whitney Duals

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A *poset*, or partially ordered set, is a set P along with a binary relation satisfying reflexivity, antisymmetry, and transitivity. For this project, we will consider finite, graded posets with a minimum element. One property of posets that has been well studied are the Whitney numbers of the first and second kind [1, 5, 6]. Recall the *Möbius function* of a poset P is defined recursively for pairs $x < y$ in P by

$$\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum\limits_{x \leq z < y} \mu(x,z) & x \neq y \end{cases} \tag{1}$$

The k^{th} *Whitney number of the first kind*, denoted $w_k(P)$ is defined as $w_k(P) = \sum\limits_{\rho(x)=k} \mu(\hat{0}, x)$ and the k^{th} *Whitney number of the second kind*, denoted $W_k(P)$ is defined by $W_k(P) = |\{x \in P | \rho(x) = k\}|$, where $\rho : P \to \mathbb{N}$ is the rank function.

Two posets, P and Q, are *Whitney Duals* if for all $k \geq 0$, $|w_k(P)| = W_k(Q)$ and $|w_k(Q)| = W_k(P)$ [3].

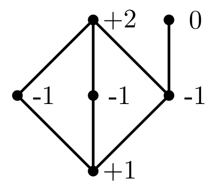

Figure 1: An example of a poset, P, which is its own Whitney dual. Notice that $|w_0(P)| = 1 = W_0(P)$, $|W_1(P)| = 3 = W_1(P)$, and $|w_2(P)| = 2 = W_2(P)$.

Whitney numbers have numerous connections to areas of combinatorics. For example, Stanley showed that Whitney numbers can be used to count acyclic orientations [6]. In [5], Whitney numbers appear as the coefficients of the chromatic polynomial of a finite graph. More recently Adiprasito, Huh, and Katz [1] proved a long standing conjecture about log-concavity of Whitney numbers of the first and second kind for geometric lattices with ideas stemming from Hodge Theory.

From the defintions above, it is clear that the k^{th} Whitney number of the second kind is far easier to compute than the k^{th} Whitney number of the first kind. Studying Whitney duals would provide tools to further compute the Whitney number of the first kind. Namely, if we know how to construct the Whitney dual of a poset P for which we are curious about $w_k(P)$, it would suffice to construct the Whitney dual Q and compute $W_k(Q)$. For GRWC 2019, I propose that we further explore Whitney Duals and, in particular, focus on the following questions:

Problem 1. *Suppose P and Q are posets with Whitney duals. Using poset operations, can we build a new poset from P and Q that has a Whitney dual?*

Björner and Wachs show that poset operations preserve lexicographic shellability, which is closely related to edge labelings [2]. In [3], edge labelings play a key role in the authors' construction of Whitney duals. We could look at how poset operations such as direct products, rank-selection, interval posets, ordinal sums, and cardinal powers affect the labeling and align our observations with Whitney duals. For example, if we take the poset  and take the direct product with itself we obtain another self Whitney Dual with $|w_0(P)| = 1 = W_0(P)$, $|W_1(P)| = 4 = W_1(P)$, and $|w_2(P)| = 4 = W_2(P)$.

The second problem involves lattices. In [4], the authors showed that every geometric lattice has a Whitney Dual. Through correspondence with Rafael González D'León, an author of [3, 4], he mentioned that a common occurance is that maximal intervals of the Whitney duals are lattices, leading to the following question:

Problem 2. *For which pairs of posets that are Whitney duals are all their maximal intervals lattices?*

In [3], the authors provide a construction of Whitney Duals based on edge labelings. One way to start Problem 2, would be to analyze the construction provided to try to determine what about the construction forms lattices in maximal intervals.

The last two problems may prove difficult, but would provide a goal in the exploration of Whitney duals.

Problem 3. *Is there a systematic way to construct a Whitney dual of P without the use of labelings [3]?*

Problem 4. *Can we count the number of non-isomorphic Whitney duals for a particular class of posets?*

A natural way to begin these problems would be to choose a class of posets such as posets of weighted partitions or non-crossing partition lattices and begin looking at the k^{th} Whitney numbers of the first and second kind; analyzing our findings with problems 3 and 4 in mind. Both of these families have been studied in [3], providing a good starting point to understand the structure of the posets in relation to Whitney duals. We can also SageMath as a tool to search for Whitney duals and generate posets.

References

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