INFINITE ODD CYCLE SATURATION GAMES

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1. Terminology and Known Results

Let$\mathcal{F}$ be a set of graphs. The $\mathcal{F}$-saturation game is a zero-sum game consisting of two players, Max and Mini. The players alternate turns adding edges to an initially empty graph G on n vertices, with the only restriction being that G is never allowed to contain a subgraph that lies in $\mathcal{F}$. The game ends when no more edges can be added, that is, when G is $\mathcal{F}$-saturated. The payoff for Max is the number of edges in G when the game ends, and Mini's payoff is the opposite of this. Thus Max wants the game to last as long as possible, while Mini wants the game to end as quickly as possible. We let $\mathrm{sat}_g(\mathcal{F}; n)$ denote the number of edges that the graph in the $\mathcal{F}$-saturation game ends with when both players play optimally, and we call this quantity the game $\mathcal{F}$-saturation number.

We note that this game, and hence the value of $\mathrm{sat}_g(\mathcal{F}; n)$, depends on whether Max or Mini makes the first move of the game, and in general this choice can significantly affect the value of $\mathrm{sat}_g(\mathcal{F}; n)$, see [4]. For simplicity we will only consider the game where Max makes the first move.

Let C_k denote the cycle of length k. The $\{C_3\}$-saturation game, also known as the triangle-free game, was the original saturation game to be studied. The first investigation into saturation games was carried out by Füredi, Reimer, and Seress [3] where they proved what is still the best known lower bound for $\mathrm{sat}_g(\{C_3\}; n)$ of $\frac{1}{2}n \log n + o(n \log n)$. Erdős claimed to have proved an upper bound of $n^2/5$ for this game, but this proof has been lost. Recently, Biró, Horn, and Wildstrom [1] published the first non-trivial asymptotic upper bound of $\frac{26}{121}n^2 + o(n^2)$ for $\mathrm{sat}_g(\{C_3\}; n)$. While little is known about the game saturation number for the triangle-free game, we were able to show that if $\mathcal{F} = \{C_3, C_5\}$, then the game saturation number is quadratic. More generally, we showed the following [7].

Theorem 1.1. *If $\mathcal{C}$ is any collection of odd cycles with $\{C_3, C_5\} \subseteq \mathcal{C}$, then,*

$$\mathrm{sat}_g(\mathcal{C}; n) \geq \frac{6}{25}n^2 + o(n^2).$$

A number of other results have been obtained for specific choices of $\mathcal{F}$, see for example [2]. In addition to this, saturation games have recently been generalized to directed graphs [5], hypergraphs [6], and to avoiding more general graph properties such as k-colorability in [4].

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2. Open Problems

There are a plethora of problems that could be looked at with regards to saturation games. For this problem proposal, we focus on saturation games where $\mathcal{F}$ consists of an infinite collection of odd cycles. To this end, we define $\mathcal{C}_\infty^o$ to be the set of all odd cycles. It isn't too difficult to show that $\mathrm{sat}_g(\mathcal{C}_\infty^o; n) = \lfloor \frac{1}{4}n^2 \rfloor$, see [2]. With this in mind, the following result is rather surprising.

Theorem 2.1. [7]

$$\mathrm{sat}_g(\mathcal{C}_\infty^o \setminus \{C_3\}; n) \le 2n - 2.$$

We believe that the constant for $\mathrm{sat}_g(\mathcal{C}_\infty^o \setminus \{C_3\}; n)$ can be determined. In particular, we conjecture the following.

Conjecture 1.

$$\mathrm{sat}_g(\mathcal{C}_\infty^o \setminus \{C_3\}; n) \sim 2n.$$

There are two natural ways to modify this game. The first is to consider the $(\mathcal{C}_\infty^o \setminus \{C_{2k+1}\})$-saturation game for various k. By Theorem 1.1 we know that this value is quadratic for $k \ge 3$ (though the exact constant is unknown), and Theorem 2.1 shows that this value is linear when $k = 1$. The value in the case $k = 2$ is completely unknown.

Problem 1. Determine bounds for $\mathrm{sat}_g(\mathcal{C}_\infty^o \setminus \{C_5\}; n)$.

Another way to modify the game would be to allow only the k smallest odd cycles to be formed. To this end we define $\mathcal{C}_{2k+1} = \{C_3, C_5, \ldots, C_{2k+1}\}$.

Problem 2. Determine bounds for $\mathrm{sat}_g(\mathcal{C}_\infty^o \setminus \mathcal{C}_{2k+1}; n)$.

One can also consider variants of all of these games by replacing sets of odd cycles with sets of even cycles, or simply by considering sets of cycles without regard to parity. We note that when avoiding infinitely many even cycles, it may make more sense to consider our "host graph" to be $K_{n,n}$ instead of K_n, see [2] for details.

References

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