# A coloring varient of the Erdős-Szekeres theorem

Josiah Reiswig

A classic result in extremal combinatorics is the Erdős-Szekeres theorem [1].

**Theorem 1.** Suppose that $k$ and $m$ are positive integers. Any sequence of real numbers of length $[km + 1]$ has an increasing sequence of length $k+1$ or a decreasing sequence of length $m + 1$.

This bound can be shown to be tight by the following permutation: $\pi = \pi_1\pi_2 \cdots \pi_k$, where

$$\begin{aligned} \pi_1 &= m, m-1, \ldots, 1 \\ \pi_2 &= 2m, 2m-1, \ldots, m+1 \\ &\vdots \\ \pi_i &= im, im-1, \ldots, (i-1)m+1 \\ &\vdots \\ \pi_k &= km, km-1, \ldots, (k-1)m+1. \end{aligned}$$

With this result in mind, we can formulate the following related question.

**Problem 1.** [2] What is the maximum number of 2-colorings of a sequence of real numbers such that no coloring admits a monochromatic, monotone subsequence of length $k + 1$?

Here, a 2-coloring of a sequence is a partition of the entries of the sequence into two color classes. We may assume that the colors are red and blue.

Let $\alpha(k)$ denote this maximum number of 2-colorings. It is worth noting that the length of the sequence realizing this maximal set of 2-colorings is not specified. For $k$ and $n$ positive integers, and $\pi$ a permutation of $[n]$, we define $c_k(n, \pi)$ to be the maximum number of 2-colorings of $\pi$ such that no coloring classes admits a monotone subsequence of length $k + 1$. Then,

$$\alpha(k) = \max_{n\geq 1} \max_{\pi\in S_n} c_k(n, \pi).$$

We can justify that this number is finite since for $n \geq 2(k^2+1)$ any 2-coloring of $[n]$ produces a color class of size $k^2+1$. By the Erdős-Szekeres theorem, this color class contains a monotone sequence of length $k + 1$.

With this bound on $n$, we easily obtain a naive upper bound for $\alpha(k)$. Since $n \leq 2(k^2 + 1)$ the number of 2-colorings of any permutation of $[n]$ is at most $2^{2k^2+1}$. Hence,

$$\alpha(k) \leq 2^{2k^2+1}.$$

As a (partial) example of calculating $c_k(n, \pi)$, suppose that $k = 2$ and $n = 5$. We consider the permutations

$$\begin{aligned} \pi_1&: \quad 1\ 2\ 3\ 4\ 5 \text{ and} \\ \pi_2&: \quad 1\ 2\ 5\ 4\ 3. \end{aligned}$$

Any 2-coloring of $\pi_1$ contains a monochromatic, increasing subsequence of length 3. So, $c_2(5, \pi_1) = 0$.

To evaluate $c_2(5, \pi_2)$, we may enumerate the valid 2-colorings without much difficulty. It is clear that no valid 2-coloring of $\pi_2$ has a color class of size 4 or 5. We may assume that 1 is colored blue. Not all three of 5, 4, and 3 can be colored red or blue. So pick one or two of those entries to be blue. This forces 2 to be colored red. Hence, there are $2 \cdot \left(\binom{3}{2} + \binom{3}{1}\right) = 12$ valid 2-colorings of $\pi_2$. We then have that $\alpha(2) \geq 12$. As we will see below, this is not a very good bound for $\alpha(2)$.

Based on the extremal construction of the original Erdős-Szekeres thoerem, Zhiyu Wang [2] provided a construction for the following lower bound of $\alpha(k)$. We consider the following permutation of $[2k^2]$. Let $\pi = \pi_1\pi_2 \cdots \pi_k$ where for each $\pi_i$,

$$\begin{aligned}
\pi_1 &= 2k, 2k-1, \ldots, 1 \\
\pi_2 &= 4k, 4k-1, \ldots, 2k+1 \\
&\vdots \\
\pi_i &= 2ik, 2ik-1, \ldots, 2(i-1)k+1 \\
&\vdots \\
\pi_k &= 2k^2, 2k^2-1, \ldots, 2(k-1)k+1.
\end{aligned}$$

Chose any 2-coloring of $\pi$ which evenly colors each subpermutation $\pi_i$. There are $\binom{2k}{k}^k$ such colorings. To illustrate this construction, the corresponding sequence for $k = 2$ is given below.

$$\pi = \underbrace{4, 3, 2, 1}_{\pi_1}, \underbrace{8, 7, 6, 5}_{\pi_2}$$

Combining these results together, we have

$$\binom{2k}{k}^k \leq \alpha(k) < 2^{2k^2+1}.$$

Should results be found for the Problem 1, we may formulate questions of saturation.

**Problem 2.** Suppose that $C$ is a set of colorings of a sequence of positive integers such that $|C| = \alpha(k) + h$ where $h$ is a positive integer. What is the minimum number of colorings in $C$ which contain a monochromatic monotonic sequence of length at least $k$?

# References


[1] L. Lovász. *Combinatorial Problems and Excercises*. North-Holland, 1979.

[2] Zhiyu Wang *Personal Communication.*