**GRWC 2019 - Computing Tutte Polynomials of Hyperplane Arrangements - Casey Pinckney**

# 1 Background

Let $k$ be a field. A **hyperplane arrangement** $\mathcal{A}$ in $k^n$ is a finite set of hyperplanes in $k^n$. We require our hyperplanes to be vector subspaces (so that they pass through the origin). A **subarrangement** $\mathcal{B}$ of $\mathcal{A}$ is a subset of the hyperplanes of $\mathcal{A}$. The **rank** of a subarrangement $\mathcal{B}$ of $\mathcal{A}$ is defined as $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \bigcap_{H \in \mathcal{B}} H$. The **intersection poset** of $\mathcal{A}$ is the partially ordered set of nonempty intersections of hyperplanes $H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$, ordered by reverse inclusion. Two hyperplane arrangements are **isomorphic** if their intersection posets are isomorphic. Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, x_2, \ldots, x_n\}$. The **graphical arrangement** corresponding to $G$ in $k^n$ is the arrangement consisting of the hyperplanes $x_i - x_j = 0$ for all $x_i x_j \in E$. The **Tutte polynomial** of a hyperplane arrangement $\mathcal{A}$ is given by

$$T_{\mathcal{A}}(x, y) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (x-1)^{r(\mathcal{A}) - r(\mathcal{B})} (y-1)^{|\mathcal{B}| - r(\mathcal{B})}$$

See [5] for a general reference on the combinatorics of hyperplane arrangements, and [1] for a more specific reference on their Tutte polynomials.

# 2 Tools

Finding closed expressions for the coefficients of Tutte polynomials is in general very difficult. One tool that has proved useful to aid in these computations is the Crapo coboundary polynomial, which is a translation of the Tutte polynomial. The coboundary polynomial and the Tutte polynomial encode the same information, but some results end up being easier and nicer to state using the coboundary polynomial. The coboundary polynomial also lends itself nicely to exponential generating function methods. See [1] for straightforward formulas to compute the Tutte polynomial from the coboundary polynomial, and vice versa.

**Definition 2.1** *Let $\mathcal{A}$ be an arrangement in $k^n$. The **coboundary polynomial** of $\mathcal{A}$ is given by*

$$\overline{\chi}_{\mathcal{A}}(q, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} q^{r(\mathcal{A}) - r(\mathcal{B})} (t-1)^{|\mathcal{B}|}$$

Ardila [1] adapted a method of Athanasiadis [2] to compute the coboundary polynomial (and thus the Tutte polynomial) by reducing to the case of working over a finite field. The finite field method assumes the hyperplane arrangement $\mathcal{A}$ is given by hyperplanes with integer coefficients (such an arrangement is called a $\mathbb{Z}$-arrangement). The equations for the hyperplanes in the $\mathbb{Z}$-arrangement can be treated as equations over the finite field $\mathbb{F}_q$ and their coefficients reduced modulo $q$. For a power $q$ of a sufficiently large prime, the reduced arrangement $\mathcal{A}_q$ is isomorphic to the original arrangement. No information is lost in this reduction (for example, linear independence of hyperplanes is preserved), and $\mathcal{A}_q$ is now an arrangement in the finite vector space ${\mathbb{F}_q}^n$. Ardila shows that because of this reduction, coboundary polynomials can be computed by counting points in the finite vector space ${\mathbb{F}_q}^n$ that share some common property. This leads to the following theorem of Ardila [1]:

**Theorem 2.2 (Ardila [1] Theorem 3.3)** *Given a $\mathbb{Z}$-arrangement $\mathcal{A}$ in $\mathbb{R}^n$ and $q$ a power of a large enough prime, it follows that*

$$q^{n - r(\mathcal{A})} \overline{\chi}_{\mathcal{A}}(q, t) = \sum_{p \in {\mathbb{F}_q}^n} t^{h(p)}$$

*where $h(p)$ is the number of hyperplanes in the reduced arrangement $\mathcal{A}_q$ on which $p$ lies.*

In the setting of graphs, the points outside of $\mathcal{A}_q$ have a similar behavior to proper graph colorings. Ardila [1] computes the coboundary polynomial for several examples of Coxeter arrangements (see Theorems 4.1, 4.2, and 4.3 [1]), making frequent use of this result and exponential generating function methods. In particular, Ardila computes the coboundary polynomial for the Coxeter arrangement of type $A_{n-1}$ in $\mathbb{R}^n$, which is defined by the hyperplanes $x_i - x_j = 0$ for all $1 \leq i < j \leq n$ (and is thus the graphic arrangement of the complete graph $K_n$). Martin and Reiner [4] compute an exponential generating function for the coboundary polynomial of the graphic matroid of the complete bipartite graph $K_{p_1,p_2}$ for primes $p_1, p_2$ using very similar techniques.

# 3 Proposed Problems

We propose the following two classes of arrangements to study. Our goal is to use the techniques described above to find exponential generating functions for the coboundary polynomials of these classes.

**Problem 1: Coboundary polynomials of 1-skeletons of cross polytopes** Let $X_{2n}$ be the complete graph $K_{2n}$ with a perfect matching removed. Then $X_{2n}$ is the 1-skeleton of a cross polytope on $2n$ vertices. For example, $X_4$ is a 4-cycle and $X_6$ is the 1-skeleton of an octahedron. The goal is to adapt the computation (Theorem 4.1 in [1]) for graphic arrangements of complete graphs to take into account the removed perfect matching, or otherwise use Theorem 3.3 in [1] and finite field methods to compute the coboundary polynomial from scratch.

**Problem 2: Coboundary polynomials of another arrangement** Edelman and Reiner [3] introduced a hyperplane arrangement $\mathcal{A}(G)$ associated to a finite graph $G$ that lies, with respect to subset containment of its hyperplanes, between the Coxeter arrangements of types $A_{n-1}$ and $B_n$. The Coxeter arrangement of type $B_n$ consists of the hyperplanes $x_i - x_j = 0$ and $x_i + x_j = 0$ for all $1 \leq i < j \leq n$, and the hyperplanes $x_i = 0$ for all $1 \leq i \leq n$. The arrangement $\mathcal{A}(G)$ is defined as follows. Given a graph $G = (V, E)$ with vertex labels $x_1, x_2, \ldots x_n$, the hyperplanes of $\mathcal{A}(G)$ are given by:

$$\mathcal{A}(G) = \{x_i - x_j = 0 \text{ for all } 1 \leq i < j \leq n,\ x_i + x_j = 0 \text{ for all } x_i x_j \in E(G),\ x_i = 0 \text{ for each loop } x_i x_i \in E(G)\}$$

The goal is to adapt previous results of [1] for the Coxeter arrangements of types $A_{n-1}$ and $B_n$ to compute an exponential generating function for the coboundary polynomial of $\mathcal{A}(G)$.

# References


[1] Federico Ardila. Computing the Tutte polynomial of a hyperplane arrangement. *Pacific J. Math.*, 230(1):1–26, 2007.

[2] Christos A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. *Adv. Math.*, 122(2):193–233, 1996.

[3] Paul H. Edelman and Victor Reiner. Free hyperplane arrangements between $A_{n-1}$ and $B_n$. *Math. Z.*, 215(3):347–365, 1994.

[4] Jeremy L. Martin and Victor Reiner. Cyclotomic and simplicial matroids. *Israel J. Math.*, 150:229–240, 2005.

[5] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.