LATTICE POLYTOPES ARISING FROM SCHUR POLYNOMIALS

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A *(convex) polytope* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$. That is,

$$\mathcal{P} = \text{conv}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\} := \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \ :, \lambda_i \geq 0 \ \text{ and } \ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Alternatively, one can express $\mathcal{P}$ as the solution set to a system $A\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^n$ and some $n \times m$ real matrix A. The *dimension* of $\mathcal{P}$, denoted $\dim(\mathcal{P})$, is the dimension of the affine subspace of $\mathbb{R}^n$ spanned by $\mathcal{P}$. Let $\mathcal{P}^\circ := \mathcal{P} \setminus \partial\mathcal{P}$ be the *relative interior* of $\mathcal{P}$. Given any polytope $\mathcal{P}$ and any $t \in \mathbb{Z}_{\geq 1}$, the tth *dilation* of $\mathcal{P}$ is $t\mathcal{P} := \{t \cdot \alpha \ : \ \alpha \in \mathcal{P}\}$. A polytope $\mathcal{P}$ is called *lattice* or *integral* if we can express $\mathcal{P}$ as the convex hull of finitely many integer points $\boldsymbol{x}_i \in \mathbb{Z}^n$ for $1 \leq i \leq k$.

There are many interesting properties regarding lattice polytopes, but we will focus on just two properties: the *integer decomposition property* and the *reflexive property*.

Definition 1. A lattice polytope $\mathcal{P}$ has the *integer decomposition property (IDP)* if given any lattice point $\boldsymbol{x} \in t\mathcal{P} \cap \mathbb{Z}^n$ for any $t \in \mathbb{Z}_{\geq 1}$, there are t lattice points $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_t \in \mathcal{P} \cap \mathbb{Z}^n$ such that $\boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_t$.

Definition 2. A polytope $\mathcal{P}$ with $\mathbf{0} \in \mathcal{P}^\circ$ is called *reflexive* if its (polar) dual polytope

$$\mathcal{P}^* := \{\boldsymbol{z} \in \mathbb{R}^n \ : \ \langle \boldsymbol{x}, \boldsymbol{z} \rangle \leq 1 \text{ for all } \boldsymbol{x} \in \mathcal{P}\}$$

is a lattice polytope.

Both of these properties are, in fact, quite rare. It is not particularly surprising that the reflexive property is rare, as the condition that $\mathcal{P}^*$ is lattice is quite restrictive on the volume and interior lattice points of $\mathcal{P}$, which one observes after computing a few small examples in low dimensions. However, the IDP intuitively feels like it should be true for all lattice polytopes. Indeed, it is true for all polytopes $\mathcal{P}$ with $\dim(\mathcal{P}) = 1, 2$, but one can construct (relatively simple) examples in any higher dimension which demonstrate failure. Subsequently, the following question is of interest when studying lattice polytopes:

Question 3. *Given a lattice polytope $\mathcal{P}$, does $\mathcal{P}$ have the IDP? Is $\mathcal{P}$ reflexive?*

The proposed project is to examine these properties within a particular family of polytopes which arises from a central object of study in algebraic combinatorics called *Schur polynomials*.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ be a partition of n, which we denote $\lambda \vdash n$. The *Young diagram* of λ is a diagram which consists of λ_i boxes in the ith row. A *semistandard Young Tableau* of λ is a filling of the Young diagram of λ with positive integers with the condition that rows are weakly increasing and columns are strictly increasing. The *Schur polynomial* of λ in m variables

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x_i^{t_i}$$

where the sum is over all semistandard Young Tableaux with fillings from the set $\{1, \ldots, m\}$ and t_i is the number of i's which appear in the filling.

Example 4. Let $\lambda = (3)$ and let $m = 3$. All of the semistandard Young Tableaux are

1	1	1		1	1	2		1	1	3		1	2	2		1	2	3		1	3	3		2	2	2		2	2	3		2	3	3		3	3	3

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Thus, the Schur polynomial is

$$s_{(3)}(x_1, x_2, x_3) = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + +x_1x_3^2 + x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3$$

Example 5. Let $\lambda = (2, 1)$ and let $m = 3$. All of the semistandard Young Tableaux are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

Thus, the Schur polynomial is

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Schur polynomials have a variety of definitions besides this combinatorial description, including a realization through a determinental process. While not obvious from the given definition, these polynomials are symmetric and are a rich source of combinatorial problems (c.f. [2]).

The polytopes arising from Schur polynomials are an example of the *Newton polytope* of a multivariate polynomial f. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ for some m and let $f(\boldsymbol{x}) = \sum_\mu c_\mu \boldsymbol{x}^\mu$ with $c_\mu = 0$ for all but finitely many $\mu \in \mathbb{Z}^m$ be a multivariate polynomial. The *Newton polytope* of f, $\mathrm{Newt}(f)$, is the polytope formed from the convex hull of the exponential vectors of f. That is

$$\mathrm{Newt}(f) := \mathrm{conv}\{\mu \,:\, c_\mu \neq 0\}.$$

We wish to consider polytopes $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$. Since Schur polynomials are homogenenous polynomials, these polytopes will be $(m-1)$-dimensional in $\mathbb{R}^m$. Subsequently, the polytopes for the Schur polynomials in Example 4 and Example 5 are 2-dimensional polytopes in $\mathbb{R}^3$. In Figure 1, we have depicted these polytopes (equivalently) in the plane for convenience.

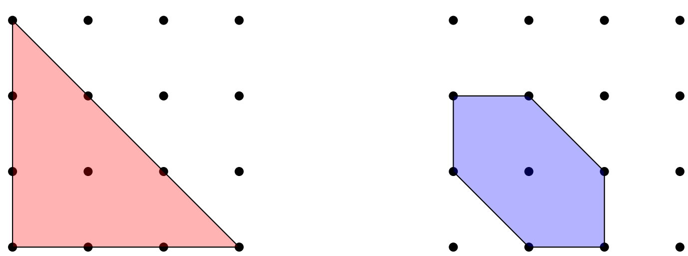

FIGURE 1. Newton polytopes for Schur polynomials in Example 4 (left) and Example 5 (right) drawn in $\mathbb{R}^2$ rather than in a 2-dimensional subspace of $\mathbb{R}^3$. Both of these polytopes can be shown to be reflexive if we translate so that the unique interior point is $(0, 0)$.

The polytopes $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$ have previously be studied by Monical, Tokcan, and Yong [4], but the following there are still many open questions. I propose the following problems:

Problem 6. *Given a $\lambda \vdash n$, when is $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$ a reflexive polytope?*

The polytope $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$ does not contain the origin in the interior, so we will consider a translation of the polytope such that this is the case. The polytope $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$ can also be realized as the $(m-1)$-dimensional polytope called the *λ-permutohedron* in $\mathbb{R}^m$ [4, 5]. That is, it is the convex hull of the S_n-orbit of $(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \in \mathbb{R}^m$. Based on some initial computations, I believe the following conjecture to be true:

Conjecture 7. *In the case that $m = n$, $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_n))$ is reflexive if and only if $\lambda = (n)$ or $\lambda = (2, 1, \ldots, 1)$.*

We can also consider other properties:

Problem 8. *Given a $\lambda \vdash n$, when does $\mathrm{Newt}(s_\lambda(x_1, \ldots, x_m))$ have the IDP?*

Computations in small dimensions have not produced an example of such a polytope without the property.

A useful tool for approaching these problems will likely be *Ehrhart theory*, which is the study of lattice point enumeration in lattice polyhedra (or more generally rational polyhedra). This theory is quite rich; one should consult [1, Chap. 2–4] and [3, Chap. 12] for background and details.

Additionally, it would be interesting to consider these questions for other polynomials of interest in algebraic combinatorics. One possible source of candidates would be *Schubert polynomials* and *Groethendieck polynomials*.

REFERENCES

[1] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. Undergraduate Texts in Mathematics. Springer, New York, 2007. Integer-point enumeration in polyhedra.

[2] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[3] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[4] Cara Monical, Neriman Tokcan, and Alexander Yong. Newton polytopes in algebraic combinatorics, 2017. arXiv:1703.02583.

[5] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

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