Strengthenings of Ryser's Conjecture

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1 Background

1.1 A graph or its complement is connected.

This is equivalent to the statement that for any 2-coloring of the edges of K_n, at least one of the monochromatic subgraphs is connected and spanning. For an r-coloring of the edges of a graph G, a monochromatic component cover is a set of monochromatic components whose union contains $V(G)$. Define the tree cover number of a graph G, $\text{tc}_r(G)$, to be the least integer k such that for any r-coloring of G, there exists a monochromatic component cover of size at most k.

In the 1970's, Ryser (see [5]) conjectured that for every r-partite hypergraph G, $\tau(G) \leq (r-1)\nu(G)$, where $\tau(G)$ is the size of a minimum vertex cover and $\nu(G)$ the size of a maximum matching. Gyárfás [4] noted that this is equivalent to the conjecture that for all G, $\text{tc}_r(G) \leq (r-1)\alpha(G)$, where $\alpha(G)$ is the independence number of G.

For the case $r = 2$, Ryser's conjecture is equivalent to Kőnig's theorem, which states that for any bipartite graph G, the size of a maximum matching is at least the size of a minimum vertex cover. Tuza [8] solved the case $r = 4$ and $\alpha(G) = 1$, and the case $r = 5$ and $\alpha(G) = 1$. In a major breakthrough, the case $r = 3$ was solved by Aharoni [1] using a version of Hall's theorem generalized to hypergraphs [2], which in turn was proved using Sperner's lemma together with special triangulations of a simplicial complex. All other cases are still open.

1.2 A graph or its complement has diameter at most 3.

This is equivalent to the statement that for any 2-coloring of the edges of K_n, there is a spanning monochromatic subgraph of diameter at most 3. In addition to the progress made by Aharoni and Tuza, progress has been made on various strengthenings of Ryser's conjecture, including finding monochromatic subgraph covers of bounded diameter. Let $D_r(G)$ be the smallest δ such that for any r-coloring of G, there exists a collection of monochromatic subgraphs of size at most $\text{tc}_r(G)$ in which each subgraph has diameter at most δ and whose union contains $V(G)$. For instance, the heading of this section is equivalent to $D_2(K_n) = 3$, the lower bound for which is given by considering a path on four vertices and its complement. To provide an upper bound δ for $D_r(G)$, one must show that in any r-coloring of the edges of G, there exist at most $\text{tc}_r(G)$ monochromatic subgraphs of diameter at most δ whose union contains $V(G)$. To provide a lower bound δ' for $D_r(G)$, one must find an r-coloring of G which requires a subgraph of diameter δ' to cover the vertices. Milićević [7] conjectured that for all $r \geq 2$, there exists a δ such that $D_r(K_n) \leq \delta$. DeBiasio, Kamel, McCourt, and Sheats [3] improved and extended the previous work of Milićević [6], [7] to give the following results.

Theorem 1

(i) $3 \leq D_2(K_{m,n}) \leq 4$.

(ii) $3 \leq D_3(K_n) \leq 4$

(iii) $2 \leq D_4(K_n) \leq 6$

(iv) *For G with $\alpha(G) = 2$,* $3 \leq D_2(G) \leq 6$

(v) $2 \leq D_3(K_{m,n}) \leq 6$

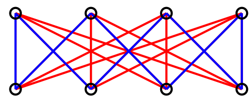

Figure 1: A 2-coloring of $K_{m,n}$, shown as a blow-up of $K_{4,4}$, in which any pair of monochromatic subgraphs which cover $V(K_{m,n})$ contains a subgraph of diameter 3.

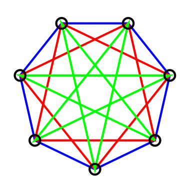

Figure 2: A 3-coloring of K_n, shown as a blow-up of K_7, in which any pair of monochromatic subgraphs which cover $V(K_n)$ contains a subgraph of diameter 3.

1.3 If a graph is not connected, then its complement is.

Another way to strengthen the conjecture that $\text{tc}_r(K_n) \leq r-1$ is to consider complete multipartite graphs. If every $(r-1)$-colored complete r-partite graph can be covered with at most $r-1$ components, then for every r-coloring of K_n, there is a component cover of size at most $r-1$ consisting of components of either only color i or only colors in $[r]\setminus\{i\}$. In particular, this would show that either K_n has at most $r-1$ red components, or we can find the component cover in the $(r-1)$-colored complete multipartite graph going between the red components. This is known [3] for $(r-1)$-colored complete r-partite graph with $r \leq 4$.

2 Problems

The above results suggest the following problems as next steps.

Problem 1 *Improve the bounds in Theorem 1.*

Lower bounds could be improved by constructing examples that require monchromatic subgraph covers of larger diameter (see Figures 1 and 2). Analyzing the methods seen in [6] and [3] could lead to insights on how to improve upper bounds. A further strengthening to consider is covering with monochromatic trees of bounded diameter instead of monochromatic subgraphs of bounded diameter.

Problem 2 *Find bounds for $D_5(K_n)$.*

Tuza [8] showed that $\text{tc}_5(K_n) = 4$. Through the work in [3], we found that a more thorough understanding of the cases of the 3-colored complete bipartite graph will be useful to better understand the diameter bound in this case.

Problem 3 *Prove or disprove: every 4-colored complete 5-partite graph has a component cover of size at most 4.*

Similar to the approach for Problem 2, we found that a better understanding of the cases of the 3-colored complete bipartite graph will be useful. In particular, understanding exactly when a 3-colored complete bipartite graph has a component cover of size at most 3, rather than 4, will allow corresponding cases of the 4-colored complete 5-partite graph to follow immediately. Furthermore, understanding the structure of the 3-colorings of the the complete bipartite graph that require 4 monochromatic components to cover the vertices will be useful in solving this problem.

References

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