On the number of disjoint cycles of specific lengths in tournaments

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1 Disjoint cycles in digraphs

A digraph $D = (V, A)$ consists of a set V of vertices and a set A of ordered pairs of vertices called arcs. If xy is an arc, we say x dominates y. The out-degree $d^+(x)$ of $x \in V$ is the number of vertices dominated by x, the in-degree $d^-(x)$ of $x \in V$ is the number of vertices dominating x and $\delta^+(D)$ and $\delta^-(D)$ are the minimum out-degree and in-degree of D, respectively.

A digraph is strong, if for any two vertices x, y, it has a path from x to y and a path from y to x. When we say graphs are disjoint, it means that no two of them have any common vertex. Cycles and paths considered here are always directed, unless otherwise specified.

One of the most famous conjecture on disjoint cycles in digraphs is the Bermond-Thomassen conjecture.

Conjecture 1.1 *([3]) Let D be a digraph. If $\delta^+(D) \geq 2k - 1$, then D contains k disjoint cycles.*

The case $k = 1$ is trivial; Thomassen [8] proved the case $k = 2$ with a very short but smart proof; it was solved by Lichiardopol, Pór and Sereni [5] for the case $k = 3$. A proof of this conjecture for tournaments was given by Bang-Jensen, Bessy and Thomassé [1] in 2014. For all other cases, the conjecture remains open.

2 Disjont cycles in tournaments

A tournament is a digraph T such that for any two distinct vertices x and y, exactly one of the ordered pairs xy and yx is an arc of T. A tournament is pancyclic, if it has cycles of all lengths from 3 to $|T|$. Tournaments possess many properties not shared by all digraphs. In 2010, Lichiardopol proposed the following conjecture strengthening Conjecture 1.1 for the case of tournaments.

Conjecture 2.1 *([4]) Let T be a tournament. If $\delta^+(T) \geq (q-1)k-1$, then T contains k disjoint cycles of length q.*

Instead of proving Conjecture 2.1, Lichiardopol [4] proved a semi-degree version of the conjecture, which is a relaxation than using the out-degree.

Theorem 2.2 *([4]) If both $\delta^+(T)$ and $\delta^-(T)$ are at least $(q-1)k-1$, then T contains k disjoint cycles of length q.*

Yan and M. [7] gave a proof of Lichiardopol's conjecture for $q \geq 11$, but there was a flaw in that proof. They fixed that flaw later. Recently they proved that the conjecture also holds when $q \leq 10$, the paper is now under preparing.

In 2018, Yan and M. improved Lichardopol's theorem by proving the following theorem.

Theorem 2.3 *([6]) Let T be a tournament and $q \geq 4$ and $k \geq 1$. If both $\delta^+(T)$ and $\delta^-(T)$ are at least $(k+2q)(3q^2-3q-4)/(6q-10)-1$, then T contains k disjoint cycles of length q. Moreover, when $q = 3$, T contains at least $16k/15-5$ disjoint triangles.*

One may check that when $k > 2q(3q^2-3q-4)/(3q^2-13q+14)$, the degree bound in Theorem 2.3 is smaller than the degree bound in Theorem 2.2.

Theorem 2.3 shows that the degree bound in Theorem 2.2 is not sharp. It is natural to ask the following problems.

Problem 2.4 *Is the degree condition on Conjecture 2.1 is sharp? If not, then what's the best possible bound to guarantee k disjoint cycles of length q under the minimum out-degree condition?*

Problem 2.5 *Does the conclusion of Theorem 2.3 hold for T with minimum out-degree?*

Bang-Jensen, Bessy and Thomassé in [1] proved that for every $\varepsilon > 0$, when k is large enough, every tournament with minimum out-degree at least $(1.5+\varepsilon)k$ contains k disjoint cycles. They also proved that the linear factor 1.5 is best possible. For general q, even the asymptotical bound is not known.

3 How to approach the problem

The method used in [6] may help to approach the problem.

Let s be the maximum number of disjoint cycles of length q in T and let $S = \{C_1, \dots, C_s\}$ be the set of disjoint cycles of length q. Write $V_1 = \cup_{1 \le i \le s} V(C_i)$ and $V_2 = V(T) - V_1$. Denote by $|V_2| = t$. Note that $T_S = T[V_2]$ is a q-cycle-free by the maximality of s. Let $P = x_1 x_2 \cdots x_t$ be a hamiltonian path of T_S. Then the following holds.

Lemma 3.1 *For i, j with $j \ge i + q - 1$, we have $(x_i, x_j) \in A(T)$.*

Lemma 3.1 implies that $\omega_i = (x_i, x_{t+1-i}) \in A(T)$ for each $1 \le i \le 3q - 5$, when V_2 is sufficiently large. Let $e(\omega_i)$ denote the number of vertices that form a triangle with ω_i and $e(\Omega_S) = \sum_{1 \le i \le 3q-5} e(\omega_i)$. Then we estimate the lower bound and the upper of $e(\Omega_S)$ and get an inequality, which gives us the bound on s. Details can be found in paper [6].

References

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