EXTENDING INTERLACE POLYNOMIAL RESULTS TO EULERIAN GRAPHS

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Let $G = (V, E)$ be a simple graph and $uv \in E$. Let V_u be the set of neighbors of u and not v, not including v, let V_v be defined analogously, and let V_{uv} be the set of vertices adjacent to both u and v. Consider the complete tripartite graph (T, E_T) with partite classes V_u, V_v, V_{uv}. The graph $G^{uv} = (V, E\Delta E_T)$ is called a pivot of G. In other words, the pivot removes all edges of G with endpoints in one of V_u, V_v, V_{uv}, but not in the same one, and adds all such edges if they were not originally in G. The interlace polynomial $q(G; x)$ of a simple graph G, introduced by [1] is defined recursively by the following.

$$q(G) = \begin{cases} q(G-u;x) + q(G^{uv}-v;x), & uv \in E(G) \\ x^n, & \text{if } G \text{ is the graph on n vertices with no edges} \end{cases}.$$

This definition does not depend on the edge chosen to pivot on. Though we will not show it here, this definition can be extended to graphs with loops.

The interlace polynomial counts the number of Eulerian circuits of a certain class of digraphs. We first provide the necessary definitions. A 2-in, 2-out digraph $D = (V, E)$ is defined to be a digraph where all vertices have in-degree 2 and out-degree 2, and hence is necessarily Eulerian. Let C be an Eulerian circuit of D. We say that vertices a, b are interlaced in C if C visits them in the sequence $a \dots b \dots a \dots b$. We define the interlace graph of D corresponding to C to be the graph $H = H(C) = (V, E_H)$ where $uv \in E_H$ if and only if u and v are interlaced in C.

Theorem 1. *Let D, C, and $H(C)$ be as above. Then $q(H(C); 1)$ gives the number of Eulerian circuits of D.*

An application of the interlace polynomial is seen in DNA sequencing [4]. Consider a de Bruijin graph, where vertices represent substrings of nucleotides, and a directed edge indicates that the suffix of the tail agrees with the prefix of the head. Then an Eulerian circuit is a way of sequencing the DNA, and the reciprocal of the total number of Eulerian circuits is the probability of reconstructing the original sequence. One basic model is to construct the graph as a 2-in, 2-out graph, where the interlace polynomial can count the number of Eulerian circuits.

Furthermore, the interlace polynomial connects to other graph polynomials. The Martin polynomial $m(D; x)$, where D is a 2-in, 2-out digraph with Eulerian circuit C, is used to study circuit partitions of such digraphs as well as 4-regular graphs, and satisfies $m(D; x) = q(H(C); x)$. Additionally, the Tutte polynomial of a planar graph can be expressed as a Martin polynomial of the corresponding medial graph, hence the interlace polynomial relates to the Tutte polynomial as well. Related to both the Martin polynomial and the interlacing polynomial is the directed circuit partition polynomial $r(D)(x)$. For a

Date: January 2019.

weakly directed Eulerian graph D, we define $r(D;x) = \sum_i r_i(D)x^i$, where $r_i(D)$ is the number of partitions of D into directed circuits. [5]

Theorem 2. *Let D, C, and $H(C)$ be defined as in Theorem (1). Then $q(H(C);x) = \frac{1}{x-1} r(D;x-1)$.*

Hence, the interlace polynomial can also be used to count circuit partitions for 2-in, 2-out digraphs.

One question raised by [5] is whether the results concerning the interlace polynomial on 2-in, 2-out digraphs can be extended to arbitrary Eulerian graphs; this can be done for Martin polynomials, which suggests the same can be done for $q(G;x)$. We would need to decide whether to consider non-simple graphs. If this fails, we could also try to find a new invariant extending to Eulerian graphs that generalizes the interlace polynomial.

Question: Can Theorems 1 and 2 be extended to arbitrary Eulerian graphs?

The proof of Theorem 1 in [1] uses the fact that by fixing an vertex v in a 2-in, 2-out digraph D, that the Eulerian circuits of D can be partitioned into two classes based on which out-edge follows which in-edge. In a general Eulerian graph, we would more classes to consider.

One method of approaching this problem is with delta-matroids, a generalization of matroids, for which a generalized interlace polynomial has been defined. Delta-matroids occur in a variety of structures, including embedded graphs, ribbon graphs, and skew-symmetric matrices [2]. They can be best understood through matroids; a matroid is a set system $M = (E, \mathcal{B})$ such that $\mathcal{B}$ is nonempty and for all distinct $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$, there exists $f \in B_2 - B_1$ such that $(B_1 - e) \cup \{f\} \in \mathcal{B}$. A delta-matroid is a set system $D = (E, \mathcal{F})$ where $\mathcal{F}$ is a nonempty collection of subsets of E, such that for all $F, G \in \mathcal{F}$, if $f \in F\Delta G$, then there exists $g \in G\Delta F$ such that $F\Delta\{f,g\} \in \mathcal{F}$. E is called the ground set, and sets in $\mathcal{F}$ are called feasible sets. We see that feasible sets generalize bases, using a symmetric exchange axiom instead.

For $X \subseteq E$, we define the distance $d_D(X)$ from X to D to be $\min\{|F\Delta X| : F \in \mathcal{F}\}$. The interlace polynomial of D is then defined to be $q_\Delta(D;x) = \sum_{T \subseteq E} x^{d_D(X)}$. For a graph G, let A be its adjacency matrix, considered over GF(2). The adjacency delta-matroid M_G of G is defined by having ground set $V(G)$, and taking the feasible sets to be all $X \subseteq V(G)$ such that the principal submatrix $A[X]$ is invertible over GF(2). Then we have $q(G;x) = q_\Delta(M_G; x-1)$. [5]

We note that it has been found that there is a two-variable generalization of the interlace polynomial, and it has been found that this polynomial in fact determines the Tutte polynomial. [6]

Finally, we mention some more difficult, but interesting conjectures from the original paper [1]. Originally, Arratia et al conjectured that the sequence of coefficients of $q(G;x)$ is unimodal (that is, the coefficients are non-decreasing up to a point, then non-increasing), but a counterexample has been found [3]. However, the counterexample found is not an Eulerian graph. It is natural to restrict this question to Eulerian graphs due to their connection with the interlace polynomial.

Question: For any Eulerian graph G, is the sequence of coefficients of $q(G;x)$ unimodal?

Question: For any weakly Eulerian graph or digraph, is the number of partitions into k circuits unimodal in k?

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