The order polytope of Young's lattice

1 Background

In [Sta86], Stanley defined the *order polytope* of a poset.

Definition 1.1. *Given a finite poset $(P, \leq)$ with $P = \{p_1, \ldots, p_n\}$, its* **order polytope** *$\mathcal{O}(P)$ is all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that*

$$\begin{array}{ll} 0 \leq x_i \leq 1 & \textit{for all } p_i \in P \textit{ and} \\ x_i \leq x_j & \textit{whenever } p_i < p_j. \end{array}$$

Given a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ with integer vertices, its m^{th} **dilate** is $m\mathcal{P} = \{mx \mid x \in \mathcal{P}\}$. Its **Ehrhart polynomial** is

$$i_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^n|$$

i.e., it is the number of integer points in the m^{th} dilate of $\mathcal{P}$. The **Ehrhart series** of $\mathcal{P}$ is

$$\mathrm{Ehr}_{\mathcal{P}}(t) = \sum_{m \geq 0} i_{\mathcal{P}}(m) t^m.$$

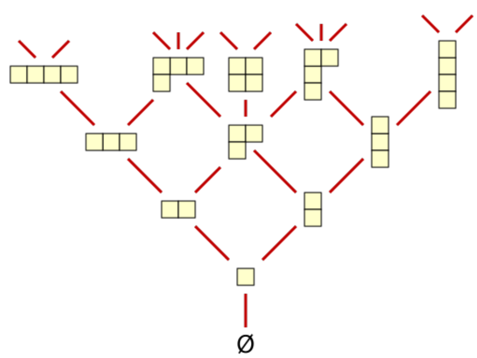
It is well known that the Ehrhart series of an n-dimensional integral polytope $\mathcal{P}$ can be written

$$\mathrm{Ehr}_{\mathcal{P}}(t) = \frac{h^*(t)}{(1-t)^{n+1}}$$

where $h^*(t)$ is known as the h^***-polynomial** of $\mathcal{P}$.

Stanley showed that the the Ehrhart polynomial of an order polytope $\mathcal{O}(P)$ is directly related to the *order polynomial* of P. The **order polynomial** of P is $\Omega_P(m)$, the number of order-preserving maps $f : P \to \{1, \ldots, m\}$. In particular, Stanley showed that $i_{\mathcal{O}(P)}(m) = \Omega_P(m+1)$ [Sta86, Theorem 4.1].

Young's lattice is the poset Y of all integer partitions ordered under inclusion of Young diagrams.



The Hasse diagram of part of Young's lattice, ~~stolen~~ used appropriately from Wikipedia.

2 Questions

Stanley's original paper has spawned numerous subsequent papers, including generalizations of order polytopes to other objects (e.g. [ABS11]) and studying order polytopes of specific classes of posets. For this workshop, the latter question is probably more approachable.

Question 1. *Let $Y_{\leq n}$ be Young's lattice restricted to partitions of $\ell \leq n$. What can be said about the combinatorial structure of $\mathcal{O}(Y_{\leq n})$? In particular, can we determine its Ehrhart polynomial or h^*-polynomial?*

There is some hope of doing this, since it has been done for other well-known posets—for example, the h^*-polynomial of the order polynomial of the "zig-zag" poset is related to permutation statistics of alternating permutations (e.g. [CS19]). Many structural aspects of Young's lattice have been studied extensively, so we will have many potential tools to use to answer Question 1.

Question 2. *Given an integer partition ν, we can answer the same questions about the order ideal in Y generated by ν?*

There is likely a connection between Question 2 and standard Young tableaux—every saturated chain in Y corresponds to a standard Young tableaux. If this fact proves useful, we could look for further connections between $\mathcal{O}(Y_{\leq n})$ and representation theory.

Question 3. *If P is the face poset of a polytope $\mathcal{P}$, can we say interesting things about $\mathcal{O}(P)$ in relation to $\mathcal{P}$? In particular, can we provide any connection between their Ehrhart and h^*-polynomials?*

A thorough literature search would be worthwhile, as many particular posets may have already been studied. (I personally could not find many such examples; aside from the aforementioned articles, [FL01] may be of interest.)

Lastly, there may be other features of $\mathcal{O}(P)$ that may be interesting to look into. For example [Sta86, Corollary 4.2] shows that $\text{vol}\mathcal{O}(P) = e(P)/n!$, where $e(P)$ is the number of linear extensions of P. Could we use this fact for the posets under consideration? Could we develop tools to study other aspects of these polytopes?

References

[ABS11] Federico Ardila, Thomas Bliem, and Dido Salazar, *Gelfand-Tsetlin polytopes and Feigin-Fourier-Littelmann-Vinberg polytopes as marked poset polytopes*, J. Combin. Theory Ser. A **118** (2011), no. 8, 2454–2462. MR 2834187

[CS19] Jane Ivy Coons and Seth Sullivant, *The h^*-polynomial of the order polytope of the zig-zag poset*, Preprint, arXiv:1901.07443, 2019.

[FL01] Stephan Foldes and Alexander Lawrenz, *On the face lattice of a poset polyhedron*, Ars Combin. **60** (2001), 313–318. MR 1849507

[Sta86] Richard P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23. MR 824105