G-Parking Functions, Spanning Trees, and the Tutte Polynomial

Carrie Frizzell

Chip-firing and Divisor Theory on Discrete Graphs

Chip-firing is a discrete dynamical system on graphs. A configuration σ of chips on a graph is an integer-valued function supported on $V(G)$. There are many versions of a chip-firing game on G. A common rule is that the configuration be positive, i.e. $\sigma(v) \geq 0$ for all v. In this case, a chip-firing move is possible if at least one vertex v has $\sigma(v) \geq val(v)$. To fire v means to give exactly one chip to each of its neighbors. A configuration is *stable* if it is not possible to fire from any vertex. In a version of chip-firing called the dollar game, described by Biggs in [Big99], σ is allowed to take a negative-value at a fixed vertex q (going into 'debt'). With this version, a configuration σ is called *recurrent* if we can get back to σ after a sequence of chip-firing moves, only firing from q if no other move is possible. The stable, reccurent configurations on a graph G form a group called the *critical group*, $K(G)$.

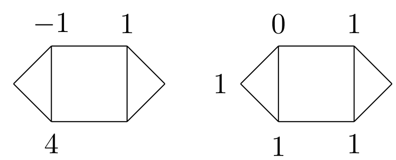


Figure 1: *The configuration σ' on the right is obtained by firing the vertex with 4 chips for the configuration σ on left.*

Let $Div(G) = \mathbb{Z}^{|V(G)|}$ be the group of $\mathbb{Z}$-linear combinations of vertices on a graph $G = (V, E)$. This is the group of divisors on G, also called configurations, as above, or abelian sandpiles. A divisor is commonly denoted

$$D = \sum_{i=1}^{n} a_i v_i.$$

Divisors D and D' are *linearly equivalent* (written $D \sim D'$) if one can be obtained from the other through a sequence of chip-firing moves. If a divisor D is equivalent to 0, it is called *principal*, and linear equivalence of divisors D and D' also means that $D - D'$ is principal.. Principal divisors can be described via the Laplacian matrix L of a finite graph. If the set of n vertices of G is ordered, consider a vector $x = (x_1, \ldots, x_n)$ with integer entries. Multiplying Lx is equated to starting from the divisor 0 and firing each vertex v_j, x_j times. This gives a divisor D, where $D(v)$ for each vertex is $(Lx)(v)$. The image of L is the group of principal divisors. The *Picard group* of G is defined as $Pic(G) = Div(G)/(D \sim 0)$, and the Jacobian $Jac(G) = Pic^0(G) = Div^0(G)/(D \sim 0)$. The degree map $deg(D) = \sum a_i$ is a surjective homomorphism from $Pic(G)$ to $\mathbb{Z}$, and we get a short exact sequence of abelian groups:

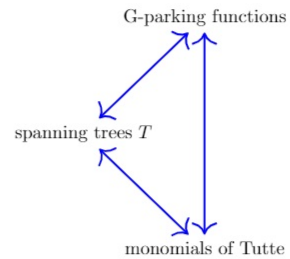
$$0 \longrightarrow Jac(G) \longrightarrow Pic(G) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Since $\mathbb{Z}$ is free (hence, projective), the sequence is split, and we get that $Pic(G) = \mathbb{Z} \bigoplus Jac(G)$. The Jacobian is a finite group, and is isomorphic to the critical group [Big99]. The order of $Jac(G)$ is the number of equivalence classes of q-reduced divisors (also called the G-parking functions); that is, for a fixed vertex q, a divisor f such that, for every subset $U \subseteq V(G) - \{q\}$, there is some $v \in U$ such that $0 \leq f(v) < outdeg_{U,G}(v)$. The value of f at q does not matter, but it will be declared that $f(q) = -1$ for convenience.

Problem Formulation

Consider the triangle diagram shown. There exist bijections represented by each of the arrows. A general question to ask is:

Q1: Under what conditions does the triangle commute?



These bijections depend on choices, such as vertex order, edge order, or base vertex. Dhar's burning algorithm [Dha90] gives a bijection from G-parking functions to spanning trees. This edge order can then be used to assign a Tutte monomial to each ST by counting the internally and externally active edges. Furthermore, an entire family of bijections between spanning trees and G-parking functions was established in [CP05], based on sets of *proper tree orders*. These include tree orders induced by vertex orderings constructed by *breadth-first*, *depth-first*, and *vertex adding* algorithms.

A vertex-order dependent bijection from G-parking functions to terms of the Tutte polynomial that bypasses spanning trees was constructed in [CMY10]. Fix a total ordering on the vertices of G. Let e be the edge between 0 and $w = min\{v \in V(G) - \{0\} | (0, v) \in E(G)\}$. Define a partition of the set of G-parking functions $\mathcal{P}$ by $\mathcal{P}_0 = \{f \in \mathcal{P} | f(w) = 0\}$ and $\mathcal{P}_1 = \{f \in \mathcal{P} | f(w) > 0\}$. Then $\mathcal{P}_0$ is in bijection with (G/e)-parking functions, and $\mathcal{P}_1$ is in bijection with $(G - e)$-parking functions [CMY10]. Here, G/e is the contraction of G relative to the edge e, and $G - e$ is G with the edge e deleted. The bijection relies on the functions π_f, where, informally, π_f records the vertices of G in the order they will burn according to a vertex-order dependent version of Dhar's Algorithm; i.e. if $\pi_f(i) = v$, this means that vertex v was the $i - th$ vertex to burn - note that fixing a vertex order tells how to break ties. We set $\pi_f(0) = q$ for base vertex q. For a fixed $f \in \mathcal{P}$ and $v \in V(G)$, denote $U = \{u \in V(G) | \pi_f^{-1}(u) \geq \pi_f^{-1}(v)\}$ and $W = \{w \in V(G) | \pi_f^{-1}(w) < \pi_f^{-1}(v)\}$.

Definition: A *critical bridge vertex* v of the parking function f with $\pi_f(i) = v$ is one for which $outdeg_{f,U}(v) = f(v) + 1$ in G (criticality), and such that there does not exist a parking function g satisfying all of the following: $g(w) = f(w)$ and $\pi_g^{-1}(w) = \pi_f^{-1}(w)$ for all $w \in W$; $g(v) = f(v) + 1$; and $\pi_g(i) > v$ in the vertex order.

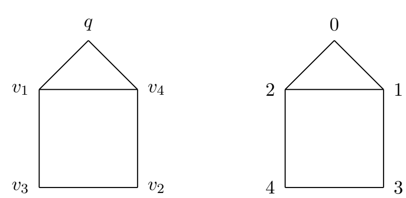


Figure 2: *Right:* The vertices labeled according to order, with base vertex q. *Left:* The vertices labeled by π_f^{-1} for $f = (-1, 1, 0, 0, 0)$. The $i - th$ component is the value at v_i. By the above definitions, v_4 is critical bridge, whereas v_1 is critical, but not bridge by comparing with $g = (-1, 2, 0, 0, 0)$.

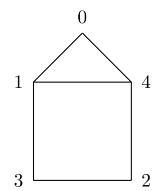
Theorem (Chang et al.): *Let $cb_f(G)$ be the number of critical bridge vertices of the function f. Then the terms of the polynomial*

$$P(G; x, y) = \sum_{f \in \mathcal{P}} x^{cb_f(G)} y^{|E(G)| - |V(G)| - \sum_{v \in V(G)} f(v)}$$

are in bijection with terms of the Tutte polynomial of G.

Q2: For a graph G, is there an edge ordering algorithm which is "natural" in that it is induced by the vertex order for the bijection *from G-parking functions to Tutte monomials*, and the triangle commutes?

The question has been answered in the negative by Zharkov and F for the G-parking function to spanning tree bijection given by Dhar's algorithm. Let G be the house graph. After writing a code to aid in computation, if the vertex order is as depicted below, then there is no total ordering on the edges $E(G)$ such that the Dhar map from G-parking functions to spanning trees and from spanning trees to monomials via the measure of internal and external activity compose to be the map from G-parking functions to monomials in [CMY10]. Suggested directions for further exploration are to check compatibility with the bijections in [CP05] and to construct a bijection from parking functions to monomials which does not depend on a vertex order. Another possible direction is to explore bijections not compatible with the proper tree orders described by Chebikin and Pylyavskyy, such as was done by Kostić and Yan in [KY08].



References

[Big99] Norman Biggs. Chip-firing and the critical group of a graph. *Journal of Algebraic Combinatorics*, 9(1): 25–45, Jan 1999.

[CMY10] Hungyung Chang, Jun Ma, and Yeong-Nan Yeh. Tutte polynomials and g-parking functions. *Advances in Applied Mathematics*, 44(3): 231 – 242, 2010.

[CP05] Denis Chebikin and Pavlo Pylyavskyy. A family of bijections between g-parking functions and spanning trees. *Journal of Combinatorial Theory, Series A*, 110(1): 31 – 41, 2005.

[Dha90] Deepak Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64: 1613–1616, Apr 1990.

[KY08] Dimitrije Kosti and Catherine H. Yan. Multiparking functions, graph searching, and the tutte polynomial. *Advances in Applied Mathematics*, 40(1): 73 – 97, 2008.