GRAPHS WITH SUBGRAPHS OF LARGE GIRTH AND CHROMATIC NUMBERS

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1 Background

In a celebrated early application of the probabilistic method, Erdős proved that there exist graphs with arbitrarily large chromatic number and girth (See [1], pp. 41-42). With this fact in mind, Erdős and Hajnal [4], [5] asked the following:

Conjecture 1 (Erdős and Hajnal 1969)**.** *For integers k and g, is there a function $f(k, g)$ such that every graph with chromatic number at least $f(k, g)$ contains a subgraph with chromatic number at least k and girth at least g?*

In the case that $g = 4$, Rödl [7] settled Conjecture 1 affirmatively. More precisely, he proved the following.

Theorem 1 (Rödl 1977)**.** *For arbitrary positive integers m, k there exists a function $\Phi(k, m)$ such that if $\chi(G) \geq \Phi(k, m)$ then G contains either*

1. *a complete subgraph K_m of order m, or*
2. *a triangle-free subgraph H with $\chi(H) \geq k$.*

Indeed Theorem 1 implies Conjecture 1 for $g = 4$; when m is large, K_m contains a triangle-free subgraph H with $\chi(H) \geq k$.

Chromatic number is often a difficult parameter with which to work. Thomassen [8] asked the following question related to Conjecture 1, but replaced the stubborn chromatic number with the friendlier average degree.

Conjecture 2 (Thomassen 1983)**.** *For integers d and g, is there a function $h(d, g)$ such that every graph with average degree at least $h(d, g)$ contains a subgraph with average degree at least d and girth at least g?*

Note that the case that $g = 4$ is easy, since a graph can be made bipartite (thereby forbidding triangles) by deleting at most half of the edges, so $f(d, 4) \leq 2d$. Conjecture 2 has been proved for $g \leq 6$ ([2], [6]).

2 Proposed Problems

The bound on $f(k, 4)$ given by Theorem 1 is quite large. In particular, $\Phi(k, 2) = 2$ for all k, and

$$\Phi(k, m) = (k - 1)^{\Phi(k,m-1)-1} + 1.$$

Since m must be large enough for K_m to contain a triangle-free subgraph of chromatic number k, this implies $f(k, 4)$ is a tower function of height at least k. The author of this result admits that he made no effort to find the smallest function that answers the question.

Question 1. *Is there a more reasonable upper bound on the function $f(k, 4)$ than the one provided in [7]?*

Confirming Conjecture 1 for all g has proven very difficult. However, the proof of Theorem 1 is only a single page. The author considers the neighborhood of each vertex and examines the adjacencies there, which is effective for forbidding C_3-s. It may be that answering the following two more-modest questions (concerning C_4-s and C_5-s) is an achievable goal, though a different method than that used in Theorem 1 will be required.

Question 2. *Is Conjecture 1 true for $g = 5$, i.e. forbidding not just triangles but also C_4-s in the desired subgraph?*

Question 3. *For any integer k, is there a function $f(k)$ such that every graph with chromatic number at least $f(k)$ contains a subgraph with chromatic number at least k that is triangle- and C_5-free?*

Question 3 corresponds to Conjecture 1 except it pertains to odd girth. It is often easier to forbid odd cycles in graphs than even cycles, so Question 3 might in fact be easier to answer than Question 2.

As an extension of Theorem 1, Rödl conjectures the following:

Conjecture 3. *For intergers k and m, is there a function $g(k, m)$ such that any graph with chromatic number at least $g(k, m)$ contains either*

1. *a complete subgraph K_m of order m, or*
2. *an* induced *triangle-free subgraph of chromatic number k?*

It is known that "triangle-free" cannot be replaced by "girth at least 5" in Conjecture 3, so this conjecture cannot be generalized to larger girths.

Conjecture 2 is true for graphs with maximum degree sufficiently "close" to their average degree. In particular it was proved in [3] that if the average degree $d(G) \geq \alpha(\log\log\Delta(G))^{\beta}$ (for some constants α, β depending on d and g), then G contains a subgraph of girth at least g and average degree at least d. The bound on $d(G)$ depends on the following lemma from [3].

Lemma 2. *Let $g \geq 3$, $d \geq 1$ be given. Suppose $\mathcal{H}$ is a simple hypergraph with $\delta(\mathcal{H})$ sufficiently large and*

$$\Delta(\mathcal{H}) \leq \delta(\mathcal{H})^{\frac{2g-1}{2g-2}}.$$

Then $\mathcal{H}$ contains a spanning subhypergraph with average degree at least d and girth at least g.

Likely this upper bound on $\Delta(G)$ in Lemma 2 can be weakened, which will immediately improve the result in [3], getting closer to a full proof of Conjecture 2.

Question 4. *Is there a smaller upper bound for $\Delta(G)$ for which Lemma 2 will still be true?*

References

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