Maker-Breaker k-AP Game

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Positional game theory is a branch of combinatorics involving the analysis of two-player games, such as the popular kids game Tic Tac Toe or abstract games played on hypergraphs. Two-player games often have the spirit that one player tries to build a specific structure while the other player tries to prevent this. Such games are called Maker-Breaker games. To be precise, a Maker-Breaker game is defined as follows.

Let (V, H) be a hypergraph. There are two players, Maker and Breaker, taking turns occupying new vertices of V. Let Maker be the player to move first. Maker wins if in the end he has occupied all vertices of a winning set $E \in H$; Breaker wins otherwise. Thus, Breaker wins if he manages to claim at least one element in every winning set.

A variety of different Maker-Breaker games have been studied. One of the most famous is the (n, q) triangle game played on the complete graph K_n. Here, Maker occupies one edge and then Breaker occupies q edges in each turn. Maker wins if he manages to occupy three edges forming a triangle; Breaker wins otherwise. When one considers the hypergraph with vertex set $E(K_n)$ and the set of triangles being the edge set, then this game fits in the previously described framework of Maker-Breaker games. The values for q for which each player has a winning strategy has been studied ([3],[4],[2]). For more information on positional games and in particular for Maker-Breaker games we refer the interested reader to the excellent survey [5].

In this research proposal, we want to study the so-called k-AP Game.

Definition 0.1. *Let $k \in \mathbb{N}$. A k-term arithmetic progression (k-AP) is a set of integers of the form $a, a + d, ..., a + (k - 1)d$ for some $a \in \mathbb{Z}$ and $d \in \mathbb{N}$.*

The (n, q) *k-AP game* is played on the integers, i.e. $V = [n]$; and the winning sets are all k-APs, i.e.

$$H = \{S \subset V \mid S \text{ forms } k\text{-AP } \}.$$

In each turn, Maker occupies one number and then Breaker occupies q numbers from $[n]$. If Maker manages to completely occupy a k-AP, he wins. Otherwise, Breaker wins. Kusch, Rué, Spiegel and Szabó proved bounds on the threshold bias for this game.

Theorem 0.2. *[1] For the (q, n) 3-AP game played on $[n]$, we have*

- *if $q \leq \sqrt{\frac{n}{12} - \frac{1}{6}}$, then Maker has a winning strategy.*
- *if $q \geq \sqrt{3n}$, then Breaker has a winning strategy.*

Proof. For the proof of the lower bound of the threshold bias, see [1]. For the upper bound on the threshold bias, let $q \geq \sqrt{3n}$. Let Breaker's strategy be to simply block all 3-APs containing Maker's last choice and one of Maker's previous choices. Maker occupies at most $\lceil n/(q+1) \rceil$ integers during the game and for each pair of integers there are at most 3 3-AP's containing them. Thus, Breaker has to block at most

$$3\left(\lceil \frac{n}{q+1} \rceil - 1\right) \leq \frac{3n}{q} \leq q$$

integers in each round to prevent Maker from completely occupying a 3-APs. □

I propose to try to close the gap between the upper and lower bounds. Clearly, Breaker's previous strategy is not optimal since it only addresses "immediate threats" without providing a strategy on which additional integers to occupy. Particularly in the beginning of the game this is quite wasteful. This strategy also does not make use the geometry of the integers; for example, the strategy does not take advantage of the fact that a pair of one odd and one even integer is only contained in two 3-APs. Already just by addressing the first issue, we can create a strategy which gives a better upper bound.

Theorem 0.3. *If $q \geq \sqrt{2.9n}$, Breaker has a winning strategy.*

Proof. We will just give a sketch of the argument ignoring floors and ceilings for simplicity.

New Strategy: Partition $[n]$ into three blocks

$$B_1 = [1, \frac{n}{3}], B_2 = [\frac{n}{3}, \frac{2n}{3}] \text{ and } B_3 = [\frac{2n}{3}, 1].$$

In every turn Breaker will block "immediate threats" caused by Maker's last turn and occupy numbers from only B_2 after that. At one point, block B_2 will be entirely occupied. W.l.o.g. let B_1 be the block among B_1 and B_3 with more occupied integers by Maker. From this point on, Breaker starts trying to block this block entirely. There will be a time when B_1 is also entirely occupied. At this point, Maker has occupied some integers of B_1. But, there cannot be a 3-AP with one number from B_1 and two numbers from B_3, so Breaker does not need to block too many 3-AP's anymore. Thus, this strategy gives an improvement. Without significant attempts to optimize the argument, the bound improves $\sqrt{2.9n}$. □

It would be interesting to figure out how good this strategy exactly is and think about better strategies.

Question 0.4. *How far can the upper bound be pushed? Which bounds can be proven for the k-AP case?*

1 References

[1] C. Kusch, J. Rué, C. Spiegel and T. Szabó (2017). *On the optimality of the uniform random strategy*. arXiv preprint arXiv:1711.07251.

[2] C. Glazik and A. Srivastav (2018). *A new Bound for the Maker-Breaker Triangle Game*. arXiv preprint arXiv:1812.01382

[3] V. Chvátal and P. Erdős. *Biased positional games*. Annals of Discrete Math 2 (1978): 221–228.

[4] J. Balogh and W. Samotij (2011). *On the Chvátal-Erdos triangle game*. the electronic journal of combinatorics, 18. Jg., Nr. P72, S. 1.

[5] M. Krivelevich. *Positional Games*. Proceedings of the International Congress of Mathematicians(ICM 2014), 4:355–379, 2014.