A two-mode network refers to a social network in which there are n persons, and m groups of which a person can be a member—these are also commonly referred to as 'agents' and 'events'. Two-mode networks were introduced in 1941 as a model to represent Southern women and the social events they attended, in order to determine the underlying social structure of the Deep South (see [1]). The data of which persons attended which events can be represented as an $n \times m$ matrix A, with $a_{i,j} = 1$ if person i participated in event j, and 0 otherwise. A two-mode network has also been used to model the ties between environmentalists and the news media, political parties and current issues, researchers collaborating on projects, and many more.

Two-mode networks can be analysed using the *direct* approach or the *conversion* approach. The direct approach refers to the analysis of the network directly through the matrix A. An obvious way to do this is to form a bipartite graph with adjacency matrix

$$\left[\begin{array}{c|c} O & A \\ \hline A^\top & O \end{array}\right].$$

This graph can then be examined through graph-theoretic techniques; usually of interest are various centrality measures. Note that a two-mode network may also be considered as a hypergraph, with A as the vertex-edge incidence matrix.

In the conversion approach, rather than examine A for structural information about the network, we consider the related matrices $AA^\top$ (where rows and columns represent agents) and $A^\top A$ (where rows and columns represent events). One benefit to the conversion approach is that some relationships between agents or between events are more easily seen from the one-mode networks provided by $AA^\top$ and $A^\top A$. For example, in the social events example, the (i, j) entry of $AA^\top$ indicates the number of events in common attended by both the i^{th} and j^{th} woman, and the (i, j) entry of $A^\top A$ represents the number of women who attended both event i and event j. Depending on which mode of the network we are more interested in analysing, we may only consider one of these *projection matrices*, although we note that information is obviously lost by analysing only one projection. Another advantage to the conversion approach is that both projection matrices are symmetric (as opposed to A) so there is a wider range of mathematical techniques available with which to analyse the network(s).

The question is posed in [2] as to whether the direct method is superior to the conversion method. In particular, they challenge the assumption that the conversion method loses important structural information about the network by analysing $AA^\top$ and $A^\top A$ alone, instead of the raw data given by A. This loss of information is referred to as "data loss", and the question is addressed by considering whether A can always be reconstructed from the projection matrices $AA^\top$ and $A^\top A$. This question is addressed more formally in [3] by considering the problem as follows: if we have $m \times n$ binary matrices A and B, such that $AA^\top = BB^\top$ and $A^\top A = B^\top B$, must it be the case that $A = B$? This question is answered in the negative, with two infinite families of such pairs produced, both highly structured. One has a connection with regular tournament matrices, and the other is connected with Ryser's notion of an interchange for a (0, 1) matrix.

The overarching question is to determine what combinatorial properties lead to a matrix exhibiting data loss. To this end, we introduce three specific questions aimed at probing this phenomenon. Note that there are also interesting algorithmic questions: can one develop an (efficient?) algorithm to determine whether a particular matrix exhibits data loss. As a first step, however, we propose the following more structural, problems.

Question. Can we determine more infinite families of pairs of matrices A, B such that $A \neq B$ and $AA^\top = BB^\top$ and $A^\top A = B^\top B$?

Note that one method that can be used to recover the matrix A from its projections is the singular value decomposition; see [2, Section 5] for explicit details. The authors observe that this reconstruction of A from $AA^\top$ and $A^\top A$ can be achieved in the case that $AA^\top$ (or $A^\top A$) does not have any repeated eigenvalues, and so our attention should be directed to cases in which there are repeated eigenvalues, such as in the examples constructed by Kirkland in [3].

Question. Suppose $A \neq B$, but $AA^\top = BB^\top$ and $A^\top A = B^\top B$. What properties of the two-mode networks determined by A and B are distinguishable/indistinguishable?

This question is a little vague, but properties of interest in two-mode networks are usually clustering coefficients and centrality measures, and they may be the most useful (in applications) to consider. A possible way to tackle this question is to frame it in terms of the direct method, and properties of the associated bipartite graph.

Question. Suppose $A \neq B$, but $AA^\top = BB^\top$ and $A^\top A = B^\top B$. Under what conditions are the bipartite graphs with adjacency matrices

$$\left[\begin{array}{c|c} O & A \\ \hline A^\top & O \end{array}\right] \text{ and } \left[\begin{array}{c|c} O & B \\ \hline B^\top & O \end{array}\right]$$

non-isomorphic?

Given the fact that a two-mode network may also be considered as a hypergraph, there are techniques from this area which may be useful in studying this problem. In particular, spectral techniques for hypergraphs are a relatively new topic which could be interesting and useful; see for example [4], where techniques are given for uniform hypergraphs. Another way to phrase the same overall question described above—removed from the applied setting and the idea of data loss—is by asking whether we can reconstruct a hypergraph from these vertex-edge incidence matrix projections, and when a unique reconstruction is not possible.

References

[1] A. Davis, B. Gardner, and M.R. Gardner. Deep South. University of Chicago Press, Chicago, 1941.

[2] M.G. Everett and S.P. Borgatti. The dual-projection approach for two-mode networks.. Social Networks, 2012.

[3] S. Kirkland. Two-mode networks exhibiting data loss. Journal of Complex Networks 6(2), 297–316. 2017.

[4] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. Linear Algebra and its Applications, 436(9), 3268-3292. 2012.