Hadamard Diagonalizable Graphs

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1 Hadamard Matrix

A *Hadamard matrix* is a ± 1 square matrix with orthogonal rows. Therefore if H is an $n \times n$ Hadamard matrix, then $HH^T = nI$, and $H^{-1} = \frac{1}{n}H^T$. It is known that if H is an $n \times n$ Hadamard matrix, then $n = 1, 2$, or $n \equiv 0 \pmod 4$. The Hadamard Conjecture [3] states that for each such n, there exists a Hadamard matrix for order n.
Given a Hadamard matrix H, the matrix obtained by permuting rows or columns of H, or by negating rows or columns of H is still a Hadamard matrix. Two Hadamard matrices are considered *equivalent* if they can be obtained from each other by permuting or negating rows or columns. Up to equivalence, there is a unique Hadamard matrix of orders 1, 2, 4, 8, and 12, and there are 5 Hadamard matrices of order 16, 3 of order 20, and 60 of order 24.
A Hadamard matrix is said to be *normalized* if all the entries in its first row and first column are 1. Any Hadamard matrix is equivalent to a normalized one.
With Sylvester's construction, a sequence of Hadamard matrices of orders 2^n is obtained. Let

$$H_1 = [1],\, H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\, H_3 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}, \cdots, H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, \cdots$$

We call H_n the *standard Hadamard matrix* of order 2^n.

2 Hadamard Diagonalizable Graphs

Let G be a graph with vertex set $V = \{1, \ldots, n\}$. Its *adjacency matrix* $A(G) = [a_{ij}]$ is the $n \times n$ matrix with $a_{ij} = 1$ if vertices i and j are adjacent, and zero otherwise. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. If $D(G)$ has constant diagonal entries, then G is called a *regular graph*. A graph G is said to be *Hadamard diagonalizable* if its Laplacian matrix $L(G)$ is diagonalizable by some Hadamard matrix. Some classes of graphs are known to be Hadamard diagonalizable.

Proposition 1. [1] *If there exists a Hadamard matrix H of order $n = 4k$ with $k \geq 1$, then K_n and $K_{2k,2k}$ are both Hadamard diagonalizable.*

Being Hadamard diagonalizable is closed under some graph operations:

Proposition 2. [1] *Let G be a Hadamard diagonalizable graph. Then $\bar{G}$ (the complement of G), $G + G$ (the union of G with itself), and $G \vee G$ (the join of G with itself) are also Hadamard diagonalizable.*
Let G_1 and G_2 be Hadamard diagonalizable graphs on m and n vertices. Then $G_1 \square G_2$ is also Hadamard diagonalizable.

Theorem 1. [1] *If G is a Hadamard diagonalizable graph, then G is regular and all its Laplacian eigenvalues are even integers.*

A graph is Hadamard diagonalizable if and only if all the eigenspace of its Laplacian matrix have a "nice" representation as a collection of ± 1 vectors. From this perspective, Proposition 1 and 2 follows almost immediately.
The complete list of Hadamard diagonalizable graphs of order 1, 2, 4, 8, and 12 is known.

Theorem 2. [1] *For $n = 2, 4, 8, 12$, the Hadamard diagonalizable graphs on n vertices are:*

1. *$n = 2$: K_2 and K_2^c (the empty graph on 2 vertices)*
2. *$n = 4$: $K_2 + K_2$, $K_{2,2}$, K_4 and K_4^c*
3. *$n = 8$: $K_2 + K_2 + K_2 + K_2$, $K_{2,2} + K_{2,2}$, $K_4 + K_4$, $(K_{2,2})\Box K_2$, K_8 and their complements*
4. *$n = 12$: $K_{6,6}$, K_{12} and their complements*

We characterized Hadamard diagonalizable graphs for a special sub-family of Hadamard matrices – the standard Hadamard matrices. It was known that cubelike graphs (the Cayley graphs on $\mathbb{Z}_2^n$ whose connection sets do not contain the zero vector) are diagonalizable by the standard Hadamard matrices [2]. We proved the converse, which allows us to state the following.

Theorem 3. [2, 4] *Let G be a graph with Laplacian matrix $L(G)$. Then $L(G)$ is diagonalizable by a standard Hadamard matrix if and only if G is a cubelike graph.*

Problem 1. *1) Characterize Hadamard diagonalizable graphs of order 16, 20, and 24.*
2) Characterize graphs that are diagonalized by some special families of Hadamard matrices, such as symmetric Hadamard matrices, and skew Hadamard matrices ($H + H^T = 2I$).
3) Characterize Hadamard diagonalizable graphs, or find more properties of Hadamard diagonalizable graphs.

The fact that Hadamard matrices of order 16, 20 and 24 are not unique up to equivalence makes the characterization a nontrivial one. Making use of the list of Hadamard diagonalizable graphs of smaller orders and Proposition 1 and 2, we can obtain some Hadamard diagonalizable graphs of these orders. The result that a graph G is Hadamard diagonalizable if and only if $L(G)$ is diagonalizable by a normalized Hadamard matrix (Lemma 4, [1]), the fact that Hadamard diagonalizable graphs are regular and have only even integer eigenvalues, and the method we use to prove that the switched 4-cube is not Hadamard diagonalizable [5] will be of use.

References

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