Suppose that $G = (V, E)$ is a hypergraph where $E \subseteq 2^V$. If we fix a colouring $\chi : V \to \{-1, +1\}$, then for each edge $e \in E$ we may define

$$\chi(e) := \sum_{v \in E} \chi(v).$$

We then refer to $\max_{e \in E} |\chi(e)|$ as the *discrepancy* of the colouring χ. The discrepancy of the hypergraph G, denoted $\text{disc}(G)$, is then defined to be the smallest attainable value amongst all colours; that is

$$\text{disc}(G) := \min_{\chi} \max_{e \in E} |\chi(e)|.$$

A long standing result from 30 years ago by Beck and Fiola [3] involves relating the discrepancy of a hypergraph to its maximum degree.

Theorem 0.1 (Beck-Fiola [3]). *If $G = (V, E)$ is a hypergraph whose maximum degree $d := \max_{v \in V} \deg(v)$, where $\deg(v) := |\{e \in E : v \in e\}|$, then*

$$disc(G) \leq 2d - 1.$$

They also famously conjectured that the exponent of the upper bound can be reduced by a half.

Conjecture 0.2 (Beck-Fiola Conjecture [3]). *If $G = (V, E)$ is a hypergraph with maximum degree $d \geq 1$ then*

$$disc(G) = O(d^{1/2}).$$

In terms of the best known upper bounds which are solely dependent on d, it has been proven that $\text{disc}(G) \leq 2d - \log^*(d)$, where $\log^*$ is the inverse of the towers of 2 function [4]. This of course yields no asymptotic improvement in terms of d.

On the other hand, if the upper bound is also allowed dependence on the hypergraph parameter $n := |V|$, then there are results yielding improvements for hypergraphs in the correct range of parameters:

Theorem 0.3 (Bansal [2]). *If $G = (V, E)$ is a hypergraph with $n = |V|$ and maximum degree $d \geq 1$, then*

$$disc(G) = O((d \log n)^{1/2}).$$

In order to find upper bounds which depend solely on the maximum degree, restricted classes of hypergraphs have instead been studied. For example, suppose that $G = (V, E)$ is assumed to be both d-regular and d-uniform; where the later term imposes that all edges of the hypergraph are of size d. In this setting, the Lovasz local lemma can be used to show that there exists a colouring with discrepancy $O((d \log d)^{1/2})$, yielding the following proposition (see [8] or [6] for details):

Proposition 0.4. *If $G = (V, E)$ is a hypergraph that is both d-regular and d-uniform, then*

$$disc(G) = O((d \log d)^{1/2}).$$

In an attempt to attain upper bounds directly matching the Beck-Fiola conjecture, random hypergraph models have been considered. The following model, which was presented by Ezra [6] in 2015, is the most commonly studied.

Consider the random hypergraph $\mathcal{G}_1$ generated as follows: Fix $n, m, d \geq 1$, and construct a set V_1 of size n to function as the vertices of $\mathcal{G}_1$. For each $v \in V_1$, choose independently and uniformly at random a subset $T_v \subseteq [m]$ of size d. We then build the edges $e_1, \ldots, e_m$, where for $i = 1, \ldots, m$

$$e_i := \{v \in V : i \in T_v\}.$$

After setting $E_1 := \{e_1, \ldots, e_m\}$, we may present the random hypergraph $\mathcal{G}_1 = (V_1, E_1)$. Observe that by definition $\mathcal{G}_1$ has maximum degree d, and for the correct range of parameters it is likely to be d-regular.

Recently, it has been show that for a wide range of parameters, namely when $d \gg \Omega(\log \log(m))^2)$, this random hypergraph matches the Beck-Fiala conjecture with high probability (denoted $w.h.p.$ in what follows):

Theorem 0.5 (Bansal [1]). *There exists constants $C_1, C_2 > 0$, such that*

$$\mathbb{P}(disc(\mathcal{G}_1) \leq C_1 d^{1/2}) = 1 - o(1),$$

provided $d \gg C_2 (\log\log(m))^2$.

While the above theorem yields an upper bound on discrepancy, it would be desirable to find a matching lower bound for a similar range of parameters.

Open Problem 0.6. *Characterize the range of parameters in n, m and d for which*

$$\mathbb{P}(disc(\mathcal{G}_1) = \Omega(d^{1/2})) = 1 - o(1)$$

In the regime in which $m = \Theta(n)$, Bansal [1] suggests that by analyzing the random incidence matrix A of $\mathcal{G}_1$, one can obtain a lower bound on $\text{disc}(\mathcal{G}_1)$ which holds *w.h.p.*. This follows as a consequence of the inequality $\text{disc}(\mathcal{G}_1) \geq \sqrt{\frac{n}{m}\lambda_n}$, where λ_n is the smallest eigenvalue of the matrix $A^T A$, and the existing literature surrounding the concentration of the spectrum of random matrices [7].

In order to see where this inequality comes from, let us define the ℓ_2-discrepancy of a hypergraph $G = (V, E)$, denoted $\text{disc}_2(G)$, where $\text{disc}_2(G) := \min_\chi \left(\frac{1}{m}\sum_{e \in E} \chi(e)^2\right)^{1/2}$. Clearly,

$$\min_\chi \left(\frac{1}{m}\sum_{e \in E} \chi(e)^2\right)^{1/2} = \min_{x \in \{-1,+1\}^n} \frac{1}{\sqrt{m}} \|Ax\|_2 .$$

Now if λ_n is the smallest eigenvalue of $A^T A$ then

$$\min_{x \in \{-1,+1\}^n} \|Ax\|_2^2 \geq \min_{\substack{x \in \mathbb{R}^n \\ \|x\|^2 = n}} x^T A^T A x \tag{1}$$

$$= n\lambda_n, \tag{2}$$

where the last line follows from the Courant-Fischer characterization of the eigenvalues of $A^T A$. Combining these inequalities with the observation that $\text{disc}(G) \geq \text{disc}_2(G)$ yields the aforementioned inequality.

While the above spectral approach is intriguing, it also seems promising that a direct analysis should work as well, perhaps for a larger range of parameters. In order to see this, consider the alternative random hypergraph model in which the same parameters are fixed as above (i.e. n, m and d):

Let us denote the random hypergraph generated from this model as $\mathcal{G}_2$. In this case, we once again fix the vertex set V_2 to be of size n. We then generate the edges $e_1, \ldots, e_m$, where for $i = 1, \ldots, m$ each $v \in V_2$ is independently placed in e_i with probability $p := d/m$. As before, we then set $E_2 := \{e_1, \ldots, e_m\}$ and return the hypergraph $\mathcal{G}_2 = (V_2, E_2)$.

Unlike the previous model, the edges are independently generated and the hypergraph is not likely to be d-regular. Analyzing the discrepancy of a colouring edge by edge therefore lends itself well to analysis, and yields the following result (proven independently):

Proposition 0.7. *If $m = \Theta(n)$, and $d \ll m$, then there exists some constant C_1 such that*

$$\mathbb{P}(disc(\mathcal{G}_2) \geq C_1 d^{1/2}) = 1 - o(1).$$

Open Problem 0.8. *Can this proposition be extended to a larger range of d? Moreover, how does the discrepancy behave in $\mathcal{G}_2$ when $m < n$? For example, in the regime when $m \ll n$ it has been shown that w.h.p $disc(\mathcal{G}_2) \leq 1$, provided $p = 1/2$ [8].*

In addition to analyzing the above models, we can also consider more standard random hypergraph models. In particular, if we introduce a new parameter $k \geq 1$ and set $m := \frac{dn}{k}$, then we can consider random k-uniform hypergraphs. For example, the binomial random k-graph $\mathbb{G}^k(n, m)$ and the d-regular k-graph $\mathbb{R}^k(n, d)$ would both be interesting to study (see [5] for detailed definitions).

Open Problem 0.9. *Asymptotically bound the discrepancy of* $\mathbb{G}^k(n, m)$ *and* $\mathbb{R}^k(n, d)$ *from above and below, where* $m = \frac{dn}{k}$*, for as large a range of parameters as possible.*

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