Flexibility of planar graphs
(suggested by Tomáš Masařík)

A *list assignment* L for a graph G is a function that to each vertex $v \in V(G)$ assigns a set $L(v)$ of colors, and an L*-coloring* is a proper coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A *request* for a graph G with a list assignment L is a function r with $\mathrm{dom}(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in \mathrm{dom}(r)$. For $\varepsilon > 0$, a request r is ε*-satisfiable* if there exists an L-coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|\mathrm{dom}(r)|$ vertices $v \in \mathrm{dom}(r)$. We say that G with the list assignment L is ε*-flexible* if every request is ε-satisfiable.

Definition 1 (ε-Flexibility [3])**.** *A graph together with a list assignment L is ε-flexible if for given ε and all possible requests there exists a proper coloring of the graph respecting L and at least an ε fraction of the request.*

First observe, that the lists are an important part of the definition. For standard k-coloring it is always possible to satisfy a $\frac{1}{k}$ fraction of any request just by permuting the colors. The notion can be easily transformed into a weighted variant by considering weighted requests.

Dvořák, Norin, and Postle [3] defined the problem and they showed some basic properties including connections with degeneracy of the graph:

Theorem 1 ([3])**.**

- *For every $d \geq 0$, there exists $\varepsilon > 0$ such that d-degenerate graphs with an assignment of lists of size $d + 2$ are weighted ε-flexible.*
- *There exists $\varepsilon > 0$ such that every planar graph with an assignment of lists of size 6 is ε-flexible.*
- *There exists $\varepsilon > 0$ such that every planar graph of girth at least 5 with an assignment of lists of size 4 is ε-flexible.*
- *There exists $\varepsilon > 0$ such that every planar graph of girth at least 12 with an assignment of lists of size 3 is ε-flexible.*

They also conjectured that their results can be improved: There exists ε such that every planar graph

a) with an assignment of lists of size 5 is ε-flexible.

b) of girth at least 4 with an assignment of lists of size 4 is ε-flexible.

c) of girth at least 5 with an assignment of lists of size 3 is ε-flexible.

Dvořák, Masařík, Musílek, and Pangrác later show Part b) in [1] and Part c) in [2] only for planar graphs of girth at least 6. We also believe that Part a) would probably require quite deep analysis since already proof of 5-choosability for general planar graphs [7] is not straightforward compared to the proof of 4-choosability for planar triangle-free graphs that is based on a degeneracy argument. Recently, it was shown [6] that planar graphs without 4-cycles with lists of size 5 are also weighted ε-flexible. In this light, I suggest to look deeper at the structure and strenghten the result further.

Conjecture 2. *There exists ε such that every planar graph with triangles but without diamonds ($K_4 - e$) with an assignment of lists of size 5 is ε-flexible.*

Another option could be to strenghten the result [6] in different direction and possibly show that for planar graphs without 4-cycles the lists of size at least 4 are sufficient to provide (weighted) ε-flexibility. This might be possible since such graphs are 4 choosable [5]. To prove the conjecture, I suggest the discharging method and, in particular, the following lemma from [3] that was already used as the main tool to prove all [1, 2, 6].

Lemma 3. *For all integers $g, k \geq 3$ and $b \geq 1$, there exists $\varepsilon > 0$ as follows. Let G be a graph of girth at least g. If for every $Z \subseteq V(G)$, the graph $G[Z]$ contains an induced $(g-3, k)$-reducible subgraph with at most b vertices, then G with any assignment of lists of size k is weighted ε-flexible.*

we say that H is a (d, k)-*reducible* induced subgraph of G if

(FIX) for every $v \in V(H)$, H is L-colorable for every assignment L such that one single vertex got just one permitted color, and

(FORB) for every d-independent set I in H of size at most $k-2$, H is L-colorable for every assignment L such that the size of the lists of vertices in I is decreased by exactly one.

As an additional orthogonal and widely-open problem, I suggest studying ε-flexibility on claw-free or perfect graphs. That area is very new since there is only a few results about flexibility in overall. Therefore the first task would be to find and formulate promising conjectures. I propose to start with the following paper [4]. They partly solve a conjecture stating that for each claw-free graph its choosability is equal to the coloring number. They show that the conjecture holds for claw-free perfect graphs with the size of maximum clique bounded by 4.

Bibliography

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