SPECIAL CASES OF THE FRANKL-FÜREDI CONJECTURE

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A λ-*intersecting family* $\mathcal{F}$ is a family of subsets of a finite set X such that any two sets in $\mathcal{F}$ have exactly λ common elements. A λ-intersecting family is called *trivial* if every set in the family shares some element. Fisher's Inequality, a fundamental result in design theory, states that if $\mathcal{F} \subset 2^X$ is a λ-intersecting family, then $|\mathcal{F}| \leq |X|$. In other words, if $\mathcal{F} \subset 2^X$ is a λ-intersecting family, then $|\partial^1 \mathcal{F}| \geq |\mathcal{F}|$, where $\partial^i \mathcal{F} := \{E \in \binom{X}{i} : E \subset F \in \mathcal{F}\}$ is the *i-shadow* of $\mathcal{F}$.

In 1991, Frankl and Füredi conjectured the following statement which implies Fisher's Inequality. One of the main motivations for this conjecture was to prove Fisher's Inequality without the use of linear algebra.

Conjecture 1. (Frankl-Füredi [4]) Let $\mathcal{F} \subset 2^X$ be a nontrivial λ-intersecting family of size m, then $|\partial^2 \mathcal{F}| \geq \binom{m}{2}$.

Several special cases of this conjecture have been proved since its publication. Before we explore these cases, we define a few important terms. A family $\mathcal{F}$ called *k-uniform* if every set in $\mathcal{F}$ has size k. A family is called *regular* if all elements of the ground set appear in the same number of sets of the family. Finally, a λ-intersecting family $\mathcal{F} \subset 2^X$ is called a *symmetric design* if $\mathcal{F}$ is uniform and if $|\mathcal{F}| = |X|$.

Frankl and Füredi [4] proved the case of Conjecture 1 when $\lambda = 1$, while Ryser [7], Woodall [9], and Babai [1] proved the case when $m = |X|$. In addition, Majindar [5] proved Conjecture 1 for regular λ-intersecting families. More recently, Chowdhury [2] proved Conjecture 1 for sufficiently small λ-intersecting families, and, in particular, that small symmetric designs satisfy the Frankl-Füredi conjecture with equality.

Theorem 1. (Chowdhury [2]) Let $\mathcal{F} \subset 2^X$ be a λ-intersecting family of size m. If

$$\sum_{F \in \mathcal{F}} \binom{|F|}{2} \geq \lambda \binom{m}{2},$$

then $|\partial^2 \mathcal{F}| \geq \binom{m}{2}$. Moreover, if $\lambda \geq 2$ and $\mathcal{F} \subset \binom{X}{k}$ is k-uniform, then $|\partial^2 \mathcal{F}| = \binom{m}{2}$ if and only if $\mathcal{F}$ is a symmetric design.

To see that the first condition necessitates a small family, consider any k-uniform family of size m. In this case, the hypothesis requires that $m \leq \frac{k(k-1)}{\lambda} + 1$.

Chowdhury [2] also proved the following special cases of the conjecture for small values of λ.

Theorem 2. (Chowdhury [2]) Let $\mathcal{F} \subset \binom{X}{k}$ be a nontrivial λ-intersecting family of size m.

(i) If $\lambda = 2$, then $|\partial^2 \mathcal{F}| \geq \binom{m}{2}$ and equality holds if and only if $\mathcal{F}$ is a symmetric design.

(ii) If $\lambda = 3$ and $k \notin \{8, 11\}$, then $|\partial^2 \mathcal{F}| \geq \binom{m}{2}$ and equality holds if and only if $\mathcal{F}$ is a symmetric design.

It is interesting to note that Stanton and Mullin [8] conjectured that if $\mathcal{F} \subset \binom{X}{k}$ is any nontrivial λ-intersecting family of size m, then $m \leq \frac{k(k-1)}{\lambda} + 1$ always, in which case the conclusion of Theorem 1 holds. Had their conjecture been true, then Theorem 2 (which follows from Theorem 1) would have implied that Conjecture 1 holds for uniform families and even characterized the case of equality.

Unfortunately, Hall [5] proved that the only k-uniform λ-intersecting families of size $m \leq \frac{k(k-1)}{\lambda} + 1$ occur when $\lambda \in \{1, 2\}$. Hall's proof when $\lambda = 2$ proves Theorem 1 for uniform, nontrivial, 2-intersecting families of size m, and since Theorem 1 and Theorem 2 (i) are equivalent for uniform, nontrivial, 2-intersecting families, we obtain Theorem 2 (i) as well.

If every 2-intersecting family satisfied the hypothesis of Theorem 1, then Conjecture 1 would be settled for $\lambda = 2$. However, one counterexample of this claim is known. Indeed, Ryser [7] showed that the following 2-intersecting family $\widehat{\mathcal{F}}$ on $X = [7]$ has size $|X|$ although $\sum_{F \in \widehat{\mathcal{F}}} \binom{|F|}{2} = 39$ and $2\binom{m}{2} = 42$.

$$\widehat{\mathcal{F}} = \{\{1,2,4\}, \{1,4,6,7\}, \{1,2,5,7\}, \{1,2,3,6\}, \{2,3,4,7\}, \{1,3,4,5\}, \{2,4,5,6\}\}$$

Note that since $m = |X|$, we know that $\widehat{\mathcal{F}}$ is not a counterexample to Conjecture 1. Recall that Babai proved Conjecture 1 when $m = |X|$. Motivated from this counterexample, Chowdhury [2] made the following conjecture.

Conjecture 2. (Chowdhury [2]) If $\mathcal{F} \subset 2^X$ is a nontrivial 2-intersecting family of size m and $\mathcal{F} \neq \widehat{\mathcal{F}}$, then

$$\sum_{F \in \mathcal{F}} \binom{|F|}{2} \geq 2\binom{m}{2}.$$

In particular, then $|\partial^2 \mathcal{F}| \geq \binom{m}{2}$.

Note that Theorem 1 implies that any uniform counterexample to Conjecture 1 also disproves Stanton and Mullin's conjecture. However, Hall's counterexample to Stanton and Mullin's conjecture is not also a counterexample for Conjecture 1. In this way, Frankl and Füredi's conjecture is weaker than that of Stanton and Mullin. Hall [5] also conjectured the following weaker claim.

Conjecture 3. (Hall [5]) For each $\lambda \in \mathbb{Z}^+$, there exists a $k_\lambda \in \mathbb{Z}^+$ such that if $k \geq k_\lambda$ and $\mathcal{F} \subset \binom{X}{k}$ is a nontrivial λ-intersecting family of size m, then $m \leq \frac{k(k-1)}{\lambda} + 1$.

I propose that we first consider Conjecture 3. Hall's conjecture and Theorem 1 together imply that Conjecture 1 holds for $\mathcal{F}$ that are k-uniform and for k sufficiently large. Some k_λ are known for small λ, but it is an open problem when $\lambda \geq 4$. I believe this question is the more approachable one and still has potential to solve more cases of the Frankl-Füredi conjecture.

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